Sufficient conditions for bipartite rigidity, symmetric completability and hyperconnectivity of graphs

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The talk is about three (families of) matroids:

- (d-dimensional) hyperconnectivity
- (d-dimensional) symmetric completion
- (d-dimensional) birigidity.

I will talk about the history of these notions, as well as some new results and conjectures.

(Joint work with Bill Jackson, Tibor Jordán and Soma Villányi.)

A little philosophy

We may study matroids on graphs because:

- they can reveal something about the graph structure,
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We may study matroids on graphs because:

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In the case of the matroids in this talk: both!

- They are natural generalizations of some well-studied graph matroids: Kalai, Kalai-Nevo-Novik, . . .
- They arise naturally in various matrix completion problems: Singer-Cucuringu, Jackson-Jordán-Tanigawa, Király-Theran-Rosen, Bernstein, . . .

For a graph G, its **incidence matrix** is the $|E| \times |V|$ matrix

$$I(G) \quad = \quad \begin{array}{c|ccccc} & & u & & v & \\ \hline uv & \cdots & 1 & \cdots & 1 & \cdots \end{array}$$

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$$I(G) \quad = \quad \begin{array}{c} \\ uv \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & u & & v & \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ \cdots & 1 & \cdots & 1 & \cdots \\ & & & & \end{array} \right) \end{array}$$

The row matroid* of $I(G)$ is the **even cycle matroid** of G.

(An edge set is independent ⇔ every connected component contains at most one odd cycle and no even cycles.)

Let us fix an orientation $\vec{G}$ of G. The **incidence matrix** of $\vec{G}$ is the $|E| \times |V|$ matrix

$$I(\vec{G}) \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} & u & & v & \end{array} \\ uv & \left(\begin{array}{ccccc} \\ \cdots & 1 & \cdots & -1 & \cdots \\ \\ \end{array} \right) \end{array}$$

Note: changing the orientation ↔ multiplying rows by -1
→ the matroid is independent of the orientation.

The row matroid of $I(\vec{G})$ is the **graphic matroid** of G.

(An edge set is independent ⇔ it is a forest.)

Observation: the row matroid of $I(\vec{G})$ comes from the pattern of the entries, not from the ± 1-s.

The **symbolic incidence matrix** of $\vec{G}$ is the $|E| \times |V|$ matrix

$$I(\vec{G}, x) \quad = \quad \begin{array}{cc} & \begin{array}{ccccc} & u & & v & \end{array} \\ uv & \left(\begin{array}{ccccc} \cdots & x_v & \cdots & -x_u & \cdots \end{array} \right) \end{array}$$

where x_v is a variable for each $v \in V$.

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where x_v is a variable for each $v \in V$.

The row matroid* of $I(\vec{G}, x)$ is (still) the **graphic matroid** of G.

But: now we can increase the dimension.

Hyperconnectivity

The 2-**dimensional hyperconnectivity matrix** of $\vec{G}$ is the $|E| \times 2|V|$ symbolic matrix

$$H_2(\vec{G},x) \quad = \quad \begin{array}{cc} & \begin{array}{ccccc} & u & & v & \end{array} \\ uv & \left(\begin{array}{ccccc} \cdots & \mathbf{x_v} & \cdots & -\mathbf{x_u} & \cdots \end{array} \right) \end{array}$$

where $\mathbf{x_v}$ is a vector of 2 variables for each $v \in V$.

Definition

The row matroid of $H_2(\vec{G},x)$ is the 2-**dimensional hyperconnectivity matroid** of G. Notation: $\mathcal{H}_2(G)$.

Hyperconnectivity

The d-**dimensional hyperconnectivity matrix** of $\vec{G}$ is the $|E| \times d|V|$ symbolic matrix

$$H_d(\vec{G}, x) \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} & u & & v & \end{array} \\ uv & \left(\begin{array}{ccccc} & & & & \\ \cdots & \mathbf{x_v} & \cdots & -\mathbf{x_u} & \cdots \\ & & & & \end{array} \right) \end{array}$$

where $\mathbf{x_v}$ is a vector of d variables for each $v \in V$.

Definition

The row matroid of $H_d(\vec{G}, x)$ is the d-**dimensional hyperconnectivity matroid** of G. Notation: $\mathcal{H}_d(G)$.

Symmetric completion

The same story with the unoriented incidence matrix:

The d-**dimensional completion matrix** of G is the $|E| \times d|V|$ symbolic matrix

$$S_d(G,x) \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} & u & & v & \end{array} \\ uv & \left(\begin{array}{ccccc} \cdots & \mathbf{x_v} & \cdots & \mathbf{x_u} & \cdots \end{array} \right) \end{array}$$

Definition

The row matroid of $S_d(G,x)$ is the d-**dimensional symmetric completion matroid** of G. Notation: $\mathcal{S}_d(G)$.

Birigidity

For a bipartite graph $G = (U, V, E)$, we have $\mathcal{H}_d(G) = \mathcal{S}_d(G)$.

(We can turn $H_d(\vec{G}, x)$ into $S_d(G, x)$ by multiplying the columns labeled by vertices in V by -1, and then multiplying some rows by -1.)

$$\begin{pmatrix} \cdots & \mathbf{x_v} & \cdots & -\mathbf{x_u} & \cdots \end{pmatrix} \rightarrow \begin{pmatrix} \cdots & \mathbf{x_v} & \cdots & \mathbf{x_u} & \cdots \end{pmatrix}$$

Definition

We call this common matroid the **d-dimensional birigidity matroid** of G. Notation: $\mathcal{B}_d(G)$.

Let G be a graph on $n \geq d+1$ vertices.

- G is d-**hyperconnected** if $\operatorname{rank}(\mathcal{H}_d(G)) = dn - \binom{d+1}{2}$.
- G is d-**completable** if $\operatorname{rank}(\mathcal{S}_d(G)) = dn - \binom{d}{2}$.

Let G be a bipartite graph on n vertices and with vertex classes of size at least d.

Fact

$$\operatorname{rank}(\mathcal{H}_d(G)) = \operatorname{rank}(\mathcal{S}_d(G)) = \operatorname{rank}(\mathcal{B}_d(G)) \leq dn - d^2$$

- G is d-**birigid** if $\operatorname{rank}(\mathcal{B}_d(G)) = dn - d^2$.

And now for something completely different

Let $M \in \mathbb{R}^{m \times n}$ be a rank-r matrix. Suppose that we “forget” some of the values in $M \to$ a partially filled matrix M'.

$$M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & * \\ * & 4 & 8 \end{pmatrix} = M'$$

Some questions:

- Can we reconstruct M from M'?
- Is there a unique solution?
- What about “generic” M?

Singer-Cucuringu 2010: a framework very similar to rigidity!

Encode the known values as a set $E \subseteq [m] \times [n]$.
Then we have the "forgetting map" $\pi_E : \mathbb{R}^{m\times n} \to \mathbb{R}^E$.

$$M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad \xrightarrow{\pi_E} \quad \begin{pmatrix} 1 & 2 & * \\ * & 4 & 8 \end{pmatrix} = M'$$

We have $M = Y^T Z$ for some $Y \in \mathbb{R}^{r\times m}$ and $Z \in \mathbb{R}^{r\times n}$. Set

$$f : \mathbb{R}^{r\times m} \times \mathbb{R}^{r\times n} \to \mathbb{R}^{m\times n} \quad (Y, Z) \mapsto Y^T Z$$

and $f_E = \pi_E \circ f$.

Idea from rigidity / differential geometry

The Jacobian matrix of f_E at (Y, Z) tells us a lot about the set $f_E^{-1}(M')$.

$$\text{Jacobian of } f_E \text{ at } (Y,Z) \;=\; \begin{array}{c} \\ ij \end{array} \begin{array}{ccccc} & i & & j & \\ \end{array} \hspace{-9em}$$

$$\begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ ij & \left(\begin{array}{ccccc} \\ \cdots & \mathbf{z_j} & \cdots & \mathbf{y_i} & \cdots \\ \\ \end{array} \right) \end{array}$$

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This is just the completion matrix for $G = ([m], [n], E)$!

$$S_r(G, x) = \begin{array}{c} \\ uv \\ \\ \end{array} \begin{array}{ccccc} & u & & v & \\ \left(\begin{array}{ccccc} & & & & \\ \cdots & \mathbf{x_v} & \cdots & \mathbf{x_u} & \cdots \\ & & & & \end{array} \right) \end{array}$$

Following this link we get:

$G = ([n],[m],E)$ is d-birigid $(\Leftrightarrow \operatorname{rank}(\mathcal{B}_d(G)) = dn - d^2)$

$$\Updownarrow$$

For generic Y, Z there are only finitely many matrices M with $\pi_E(M) = \pi_E(Y^T Z)$.

There are similar connections between

- the d-dimensional symmetric completion matroid $\mathcal{S}_d(G)$ and rank-d **symmetric/PSD** matrix completion,
- the $2d$-dimensional hyperconnectivity matroid $\mathcal{H}_{2d}(G)$ and rank-$2d$ **skew-symmetric** matrix completion.

Let’s focus on birigidity

We say that G is **$\mathcal{B}_d$-independent** if $\mathcal{B}_d(G)$ is a free matroid.

What is known:

- $\mathcal{B}_1$-independent graphs = forests.
- No “good” $(= \mathrm{NP} \cap \textsc{co}\text{-}\mathrm{NP})$ characterization is known for $\mathcal{B}_d$-independence when $d \geq 2$; but:
- G is $\mathcal{B}_2$-independent $\Leftrightarrow$ it has an acyclic orientation with no alternating cycles. (Bernstein)
- Every planar bipartite graph is $\mathcal{B}_2$-independent. (Kalai, Nevo, Novik)
- $\mathcal{B}_d$-independent graphs are $\mathcal{R}_d$-independent. (Crespo Ruiz, Santos)

Theorem (Villányi 2025)

Every $d(d+1)$-connected graph is d-rigid.

Theorem (G-Jackson-Jordán-Villányi 2025+)

Every $(10^5 d^3)$-connected bipartite graph is d-birigid.

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- $\mathcal{B}_d$-seeds: highly structured subgraphs.

Definition

Given a bipartite graph G, a subgraph of G is a $\mathcal{B}_d(G)$**-seed** if it is the $(d+1)$-core of a basis of $\mathcal{B}_d(G)$.

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- $\mathcal{B}_d$-seeds: highly structured subgraphs.
- k-biconnectivity: a new notion of vertex-connectivity for bipartite graphs.

Definition

A bipartite graph $G = (U, V, E)$ is **k-biconnected** if $|U|, |V| \geq k$ and G has no vertex cut $C \subseteq (U \cup V)$ with $|C \cap U| \leq k - 1$ and $|C \cap V| \leq k - 1$.

Theorem (G-Jackson-Jordán-Villányi 2025+)

Every $(10^5 d^3)$-connected bipartite graph is d-birigid.

Many new ideas in the proof:

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Theorem (G-Jackson-Jordán-Villányi 2025+)

Every $(10^5 d^3)$-connected bipartite graph is d-birigid.

Many new ideas in the proof:

- $\mathcal{B}_d$-seeds: highly structured subgraphs.
- k-biconnectivity: a new notion of vertex-connectivity for bipartite graphs.
- Key graph operation: double 1-extension.

Theorem (G-Jackson-Jordán-Villányi 2025+)

Every $(10^5 d^3)$-connected bipartite graph is d-birigid.

Conjecture (G-Jackson-Jordán-Villányi 2025+)

Every $(2d^2)$-connected bipartite graph is d-birigid.

A nonbipartite extension for $\mathcal{H}_d$ and $\mathcal{S}_d$:

Conjecture (G-Jackson-Jordán-Villányi 2025+)

There exists a positive integer k_d such that every k_d-connected graph G satisfies $\operatorname{rank}(\mathcal{H}_d(G)) \geq dn - d^2$ and $\operatorname{rank}(\mathcal{S}_d(G)) \geq dn - d^2$.

Thank you!

Some reading:

- Garamvölgyi, Jackson, Jordán, Villányi, **Sufficient conditions for bipartite rigidity, symmetric completability and hyperconnectivity of graphs**, 2025. arXiv:2511.00298
- Kalai, **Hyperconnectivity of graphs**, *Graphs and Combinatorics*, 1985.
- Singer, Cucuringu, **Uniqueness of Low-Rank Matrix Completion by Rigidity Theory**, *SIAM Journal on Matrix Analysis and Applications*, 2010.