Sparsity, stress-independence and globally linked pairs in graph rigidity theory

(Joint work with Bill Jackson and Tibor Jordán)

Dániel Garamvölgyi (HUN-REN-ELTE Egerváry Research Group, Budapest)

Rigidity, Flexibility and Complexity of Geometric Constraint Systems

Chennai Mathematical Institute, Chennai, January 21, 2026

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Theorem (G-Jackson-Jordán 2025+)

Let $G = (V, E)$ be a graph on at least three vertices, and let $u, v \in V$. If $\{u, v\}$ is d-linked in $G - z$ for all $z \in V - \{u, v\}$, then $\{u, v\}$ is globally d-linked in G.

I will talk about

- what this theorem says,
- its proof,
- a corollary.

Recall: definition of rigid d-frameworks and d-rigid graphs.

A d-framework (G, p) is **globally rigid** if every d-framework (G, q) satisfying

$$\|q(x) - q(y)\| = \|p(x) - p(y)\| \;\; \forall xy \in E(G)$$

is congruent to (G, p).

Theorem (Gortler-Healy-Thurston 2010)

If some generic d-framework of G is globally rigid, then all generic d-frameworks of G are globally rigid.

That is, global rigidity of d-frameworks is a **generic property**.

Globally rigid graphs

Definition

A graph is **globally d-rigid** if its generic d-frameworks are globally rigid.

We know:

- G is globally 1-rigid $\Leftrightarrow$ 2-connected
- G is globally 2-rigid $\Leftrightarrow$ 3-connected and $G - e$ is 2-rigid for every $e \in E(G)$
- G is globally d-rigid $\Rightarrow$ d-connected and $G - e$ is d-rigid for every $e \in E(G)$

The theorem of Tanigawa

Theorem (Tanigawa 2015)

Let $G = (V, E)$ be a graph on at least three vertices.
If $G - z$ is d-rigid for all $z \in V$, then G is globally d-rigid.

Our theorem is a **local** version of this.

First, we need to define the local versions of rigidity and global rigidity.

Linked vertex pairs

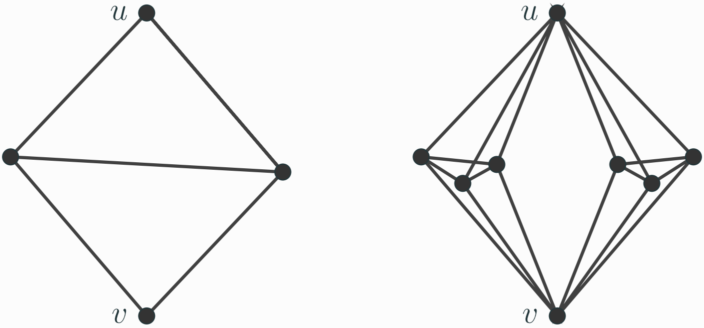
Let (G, p) be a generic framework. Then

$$(G,p) \text{ is rigid} \Leftrightarrow \text{every continuous motion preserves all the distances } \|p(u) - p(v)\|, u, v \in V.$$

Now fix $u, v \in V$.

Definition

The pair $\{u, v\}$ is **linked** in (G, p) if every continuous motion of (G, p) preserves the distance $\|p(u) - p(v)\|$.


u
v
u
v

Definition

The pair $\{u, v\}$ is **linked** in (G, p) if every continuous motion of (G, p) preserves the distance $\|p(u) - p(v)\|$.

Fact

This is a **generic property**: if $\{u, v\}$ is linked in some generic d-framework of G, then it is linked in all of them.

Definition

The pair $\{u, v\}$ is d-**linked** in G if $\{u, v\}$ is linked in all generic d-frameworks of G.

Definition

The pair $\{u, v\}$ is **globally linked** in (G, p) if for every d-framework (G, q) satisfying

$$\|q(x) - q(y)\| = \|p(x) - p(y)\| \;\; \forall xy \in E(G)$$

we have

$$\|q(u) - q(v)\| = \|p(u) - p(v)\|$$

Sad fact

This is a **not a generic property**!

Definition

The pair $\{u, v\}$ is **globally d-linked** in G if $\{u, v\}$ is globally linked in all generic d-frameworks of G.

So for a graph $G = (V, E)$

- G is d-rigid $\Leftrightarrow$ $\{u, v\}$ is d-linked in G for all $u, v \in V$,
- G is globally d-rigid $\Leftrightarrow$ $\{u, v\}$ is globally d-linked in G for all $u, v \in V$.

Mantra

Statements about rigid/globally rigid graphs often have a **local version** about linked/globally linked pairs.

Vertex-redundantly linked theorem

Theorem (Tanigawa 2015)

Let $G = (V, E)$ be a graph on at least three vertices.
If $G - z$ is d-rigid for all $z \in V$, then G is globally d-rigid.

Vertex-redundantly linked theorem

Theorem (Tanigawa 2015)

Let $G = (V, E)$ be a graph on at least three vertices.
If $G - z$ is d-rigid for all $z \in V$, then G is globally d-rigid.

Theorem (G-Jackson-Jordán 2025+)

Let $G = (V, E)$ be a graph on at least three vertices, and let $u, v \in V$. If $\{u, v\}$ is d-linked in $G - z$ for all $z \in V - \{u, v\}$, then $\{u, v\}$ is globally d-linked in G.

Proof difficulties

Theorem (Tanigawa 2015)

Let $G = (V, E)$ be a graph on at least three vertices.
If $G - z$ is d-rigid for all $z \in V$, then G is globally d-rigid.

The proof by Tanigawa relies crucially on the fact that global rigidity of d-frameworks is a generic property.

Since being globally linked is not a generic property, we need a completely new argument.

Proof ingredients

Theorem (G-Jackson-Jordán 2025+)

Let $G = (V, E)$ be a graph on at least three vertices, and let $u, v \in V$. If $\{u, v\}$ is d-linked in $G - z$ for all $z \in V - \{u, v\}$, then $\{u, v\}$ is globally d-linked in G.

Our proof uses:

- a **stress-based** sufficient condition for globally linked pairs,
- some tricks related to equilibrium stresses.

It is quite short! (roughly 1 page)

Stresses

Let

- (G, p) be a d-framework, and
- $\omega = (\omega_{uv})_{uv \in E} \in \mathbb{R}^E$ a vector of edge weights.

Definition

We say that ω is a **stress** of (G, p) if the following **equilibrium conditions** are satisfied for every vertex $v \in V(G)$:

$$\sum_{u:uv \in E} \omega_{uv}(p(u) - p(v)) = 0.$$

Stresses and linked vertex pairs

For a nonadjacent pair of vertices $u, v \in V(G)$ and a generic framework (G, p), we have

$\{u, v\}$ is linked in (G, p)

$\Updownarrow$ (definition)

every continuous motion of (G, p) preserves the distance $\|p(u) - p(v)\|$,

$\Updownarrow$

$(G + uv, p)$ has a stress ω such that $\omega_{uv} \neq 0$.

Stresses and globally linked vertex pairs

Definition (G 2023+)

A vertex pair $\{u, v\}$ is **d-stress-linked** in G if

- $\{u, v\}$ is d-linked in G, and
- For every generic (G, p) and every (G, q), if (G, p) and (G, q) have the same stresses, then $(G + uv, p)$ and $(G + uv, q)$ also have the same stresses.

Theorem (G 2023+)

If $\{u, v\}$ is d-stress-linked in G, then it is globally d-linked in G.

Upgraded theorem

Theorem (G 2023+)

If $\{u, v\}$ is d-stress-linked in G, then it is globally d-linked in G.

We actually prove:

Theorem (G-Jackson-Jordán 2025+)

Let $G = (V, E)$ be a graph on at least three vertices, and let $u, v \in V$. If $\{u, v\}$ is d-linked in $G - z$ for all $z \in V - \{u, v\}$, then $\{u, v\}$ is d-stress-linked in G.

A key lemma

Key observation: we do not need to check the equilibrium conditions at every vertex to show that ω is a stress of (G, q).

Lemma

Let (G, p) and (G, q) be d-frameworks, where (G, p) is in general position, and let $\omega \in \mathbb{R}^E$ be a stress of (G, p). If ω is not a stress of (G, q), then there are at least $d + 2$ vertices where the equilibrium conditions are violated.

An application: dimension dropping

Tanigawa's theorem + **coning** implies:

Theorem (Jordán 2017)

If G is $(d+1)$-rigid, then it is globally d-rigid.

An application: dimension dropping

Tanigawa's theorem + **coning** implies:

Theorem (Jordán 2017)

If G is $(d+1)$-rigid, then it is globally d-rigid.

The vertex-redundantly linked theorem + coning implies:

Theorem (G-Jackson-Jordán 2025+)

Let $G = (V, E)$ and $u, v \in V$. If $\{u, v\}$ is $(d+1)$-linked in G, then it is globally d-linked in G.

Thank you!

Some papers:

- Tanigawa, **Sufficient conditions for the global rigidity of graphs**, *JCTB*, 2015.
- Garamvölgyi, Jackson, Jordán, **Sparsity, Stress-Independence and Globally Linked Pairs in Graph Rigidity Theory**, 2025, *arxiv:2509.03150*.
- Garamvölgyi, **Stress-linked pairs of vertices and the generic stress matroid**, 2023 (revised in 2025), *arxiv:2308.16851*.