Stable cuts, planar rigidity, and global rigidity on the line

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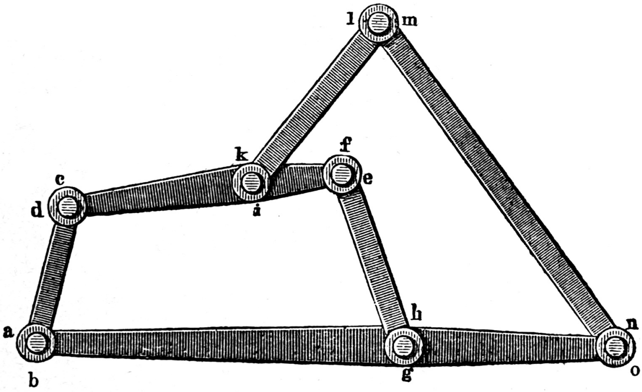
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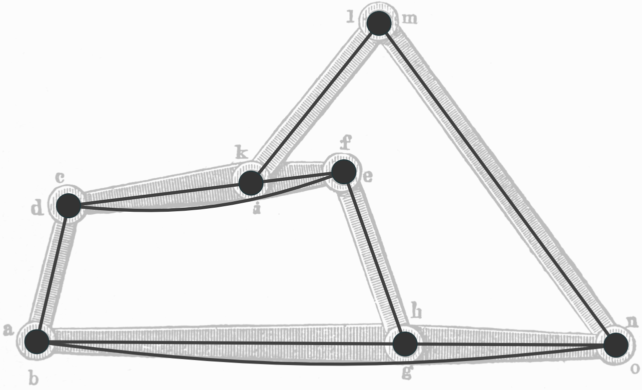

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Frameworks and rigidity

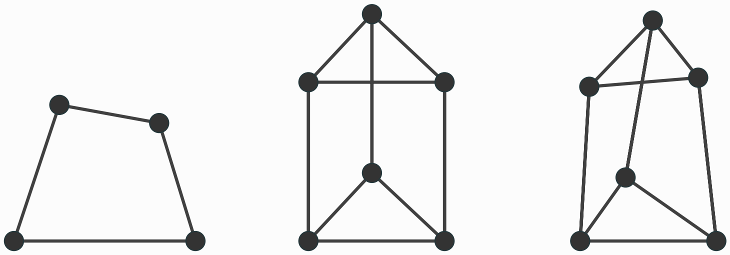
Definition

A (bar-joint) **framework** (in $\mathbb{R}^d$) is a pair (G, p), where G is a graph and p maps the vertices of G into $\mathbb{R}^d$.

Definition

A framework (G, p) is **flexible** if it has a continuous edge-length preserving deformation; otherwise, (G, p) is **rigid**.

Some examples



Combinatorics of rigidity

Main combinatorial question:

> For which graphs G do we have that "almost all" frameworks (G, p) in $\mathbb{R}^d$ are rigid?

$\rightarrow$ interesting graph classes with very nice combinatorial structure!

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What about the following question?

For which graphs G do we have that all* frameworks (G, p) in $\mathbb{R}^d$ are rigid?

*we assume that there are no zero-length edges!

One hard case

Let G be a graph.

- There is a flexible framework of G in $\mathbb{R}^1$
  $\iff$ G is **disconnected**.
  ($\iff$ every framework of G in $\mathbb{R}^1$ is flexible.)

- For $d \geq 3$, there is a flexible framework of G in $\mathbb{R}^d$
  $\iff$ G is **not a complete graph**.

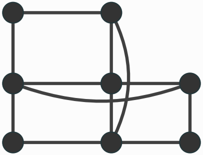
$\rightarrow$ the only interesting case is when $d = 2$.

Grid-like frameworks

Definition

A framework (G, p) in $\mathbb{R}^2$ is **grid-like** if every edge is either horizontal or vertical. A grid-like framework is **non-trivial** if it does not lie on a line.

Observation: a non-trivial grid-like framework is always flexible.



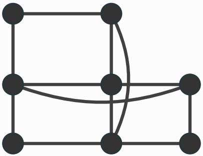
The flexibility theorem

Theorem (Grasegger-Legerský-Schicho 2019)

The following are equivalent for a graph G.

(a) G has a flexible framework in $\mathbb{R}^2$.

(b) G has a non-trivial grid-like framework in $\mathbb{R}^2$.



The flexibility theorem

Theorem (Grasegger-Legerský-Schicho 2019)

The following are equivalent for a graph G.

(*a*) G has a flexible framework in $\mathbb{R}^2$.

(*b*) G has a non-trivial grid-like framework in $\mathbb{R}^2$.

(*c*) G has a **NAC-coloring**: the edges of G can be colored with exactly two colors such that every cycle is either monochromatic, or contains at least two edges of both colors.

Computational troubles

The coloring condition implies that deciding whether a graph has a flexible framework in $\mathbb{R}^2$ is in NP (has an efficiently verifiable witness).

Theorem (G 2022)

Deciding whether a graph has a flexible framework in $\mathbb{R}^2$ is NP-complete.

Theorem (Laštovička-Legerský 2024+)

It remains NP-complete even for graphs G satisfying $|E(G)| \leq (2+\varepsilon)|V(G)|$, for every $\varepsilon > 0$.

Laman graphs

Definition

A graph G on n vertices is a **Laman graph** if it has $2n-3$ edges and for all $k \geq 2$, every k-vertex subgraph of G has at most $2k-3$ edges.

A graph is a **2-tree** if it can be obtained by repeatedly gluing triangles along edges.

Flexible realizations of Laman graphs

Conjecture (Grasegger-Legerský-Schicho 2019)

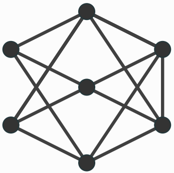
Let G be a Laman graph on at least 3 vertices. Then G has a flexible framework in $\mathbb{R}^2$ if and only if G is not a 2-tree.

Flexible realizations of Laman graphs

Theorem (Clinch-G-Haslegrave-Hyunh-Legerský-Nixon 2024+)

Let G be a Laman graph on at least 3 vertices. Then G has a flexible framework in $\mathbb{R}^2$ if and only if G is not a 2-tree.

Key idea: try to find a **stable cuts**, a set of mutually nonadjacent vertices whose removal disconnects the graph.



Flexible realizations of Laman graphs

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Let G be a Laman graph on at least 3 vertices. Then G has a flexible framework in $\mathbb{R}^2$ if and only if G is not a 2-tree.

Key idea: try to find a **stable cuts**, a set of mutually nonadjacent vertices whose removal disconnects the graph.

- If G contains a stable cut, then it has a flexible framework in $\mathbb{R}^2$. (Map the cut vertices to the same point in space, rotate around this point.)
- There is a combinatorial characterization of graphs on n vertices and $2n-3$ edges that contain a stable cut.

References

Thank you!

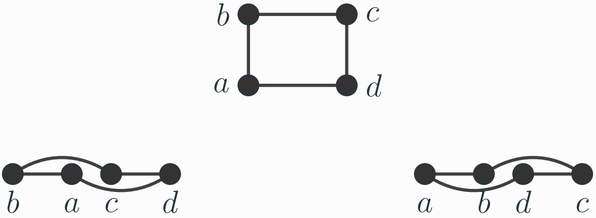
Some references:

- Clinch et al., **Stable cuts, NAC-colourings and flexible realisations of graphs**, 2024. *arXiv:2412.16018*
- Garamvölgyi, **Global rigidity of (quasi-)injective frameworks on the line**, *Discrete Mathematics*, 2022.
- Grasegger, Legerský, Schicho, **Graphs with Flexible Labelings**, *Discrete & Computational Geometry*, 2019.
- Laštovička, Legerský, **Flexible realizations existence: NP-completeness on sparse graphs and algorithms**, 2024. *arxiv:2412.13721*

Grid-like frameworks revisited

Simple observation

Non-trivial grid-like frameworks can be collapsed onto the line in two different ways.



When constrained to lie on a line, these frameworks are rigid, but not **globally rigid**.

Global rigidity on the line

Theorem (G 2022)

The following are equivalent for a graph G.

(*a*) G has a framework in $\mathbb{R}^1$ that is not globally rigid.

(*b*) G has a non-trivial grid-like framework in $\mathbb{R}^2$.

(*c*) G has a flexible framework in $\mathbb{R}^2$.