On the rigidity matroid of highly connected graphs

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Reconstruction from the graphic matroid

Let G be a graph, and let $\mathcal{M}(G)$ denote its graphic matroid.

Theorem (Whitney 1933)

If G is 3-connected, then $\mathcal{M}(G)$ uniquely determines G.

This means that (in principle) we can reconstruct G from $\mathcal{M}(G)$.

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In this talk I will

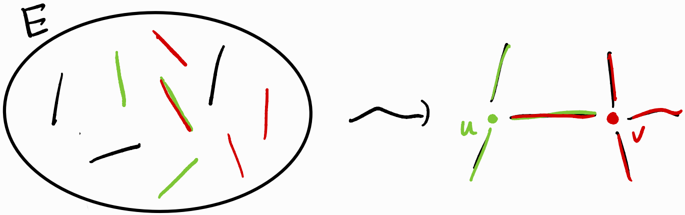
- explain this result, and
- describe generalizations to other families of matroids associated to graphs.

Identifying vertex stars in the graphic matroid

Let $G = (V, E)$ and $\mathcal{M}(G) = (E, r)$.

Key observation

To reconstruct G from $\mathcal{M}(G)$, it suffices to identify which subsets of the ground set correspond to vertex stars.



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Key observation

To reconstruct G from $\mathcal{M}(G)$, it suffices to identify which subsets of the ground set correspond to vertex stars.

- If $F \subseteq E$ is a connected* hyperplane** in $\mathcal{M}(G)$, then $E - F$ is a vertex star in G.
- If G is 3-connected, then the complement of any vertex star in G is a connected hyperplane of $\mathcal{M}(G)$.

nontrivial

*connected: for any bipartition $F = F_1 \cup F_2$, $r(F_1) + r(F_2) \geq r(F) + 1$.

**hyperplane: F is closed and has rank $r(F) = r(E) - 1$.

Graphic matroid facts for the proof

Thus in a 3-connected graph G, we have

(complements of) vertex stars in G

$$\Updownarrow$$

connected hyperplanes of $\mathcal{M}(G)$.

This follows easily from:

- G is 2-connected $\Leftrightarrow$ $\mathcal{M}(G)$ is connected.
- $\mathcal{M}(G)$ is connected $\Rightarrow \operatorname{rank}(\mathcal{M}(G)) = |V(G)| - 1$
  $\rightarrow$ the number of vertices is determined by $\mathcal{M}(G)$!

Theorem (Whitney 1933)

If G is 3-connected, then $\mathcal{M}(G)$ uniquely determines G.

In the rest of the talk, I will describe analogues for

- (k, ℓ)-count matroids,
- the C_2^1-cofactor matroid,
- the d-dimensional generic rigidity matroid,
- the very general setting of *graph matroid families*.

Count matroids

Count matroids

Fix integers k, ℓ with $k \geq 1$ and $\ell \leq 2k-1$.

A graph G is (k,ℓ)-**sparse** if every subgraph $H = (V', E')$ satisfies $|E'| \leq k|V'| - \ell$.

The (k,ℓ)-**count matroid** $\mathcal{M}_{k,\ell}(G)$ of G is defined by

$$E' \subseteq E \text{ is independent in } \mathcal{M}_{k,\ell}(G)$$

$$\Updownarrow$$

$$E' \text{ induces a } (k,\ell)\text{-sparse subgraph of } G.$$

Some examples of count matroids

Count matroids include some familiar matroids:

- $\mathcal{M}_{1,1}(G) =$ the graphic matroid of G,
- $\mathcal{M}_{1,0}(G) =$ the bicircular matroid of G,
- $\mathcal{M}_{k,k}(G) =$ the k-fold union of the graphic matroid of G,
- $\mathcal{M}_{2,3}(G) =$ the 2-dimensional generic rigidity matroid of G.

There are efficient algorithms available for computing the rank function of $\mathcal{M}_{k,\ell}(G)$.

Reconstruction from count matroids

Fix integers $k, \ell > 0$ with $\ell \leq 2k-1$, and set $c = \max(2k, 2\ell)$.

We have the following generalization of Whitney's theorem for the (k, ℓ)-count matroid.

Theorem (Jordán-Kaszanitzky 2013, G-Jordán-Király 2024)

If G is $(c+1)$-connected, then $\mathcal{M}_{k,\ell}(G)$ uniquely determines G.

Identifying vertex stars in count matroids

Fix integers $k, \ell > 0$ with $\ell \leq 2k-1$, and set $c = \max(2k, 2\ell)$.

As before, it is enough to identify the vertex stars in $\mathcal{M}_{k,\ell}(G)$.

We show that

- If $\mathcal{M}_{k,\ell}(G)$ is connected and $F \subseteq E$ is a connected k-hyperplane* in $\mathcal{M}_{k,\ell}(G)$, then $E - F$ is a vertex star in G.
- If G is $(c+1)$-connected, then $\mathcal{M}_{k,\ell}(G)$ is connected and the complement of any vertex star in G is a connected k-hyperplane of $\mathcal{M}_{k,\ell}(G)$.

* k-hyperplane: F is closed and has rank $r(F) = r(E) - k$.

Count matroid facts for the proof

Thus in a $(c+1)$-connected graph G we have

(complements of) vertex stars in G

$\Updownarrow$

connected k-hyperplanes of $\mathcal{M}_{k,\ell}(G)$.

This follows easily from:

Theorem (G-Jordán-Király 2024 + others)

- G is c-connected $\Rightarrow$ $\mathcal{M}_{k,\ell}(G)$ is connected.
- $\mathcal{M}_{k,\ell}(G)$ is connected $\Rightarrow$ $\mathrm{rank}(\mathcal{M}_{k,\ell}) = k|V(G)| - \ell$
  $\rightarrow$ the number of vertices is determined by $\mathcal{M}_{k,\ell}(G)$!

The C_2^1-cofactor matroid

The C_2^1-cofactor matroid

For a graph G, the C_2^1-**cofactor matroid** $\mathcal{C}_2^1(G)$ of G is defined as the row matroid of a certain symbolic matrix.

$$v \in V(G) \rightsquigarrow x_v, y_v,$$

$$uv \in E(G) \rightsquigarrow D(u,v) = \left((x_u - x_v)^2, (x_u - x_v)(y_u - y_v), (y_u - y_v)^2\right),$$

$$\begin{array}{cc} & \begin{array}{ccccc} \cdots & u & \cdots & v & \cdots \end{array} \\ \begin{array}{c} \vdots \\ uv \\ \vdots \end{array} & \begin{pmatrix} & & & & \\ 0 & D(uv) & 0 & -D(uv) & 0 \\ & & & & \end{pmatrix} \end{array}$$

The C_2^1-cofactor matroid

For a graph G, the **C_2^1-cofactor matroid** $\mathcal{C}_2^1(G)$ of G is defined as the row matroid of a certain symbolic matrix.

Recently, Clinch, Jackson and Tanigawa gave a combinatorial formula for its rank function.

Theorem 2.5. *[3, Theorem 6.1] Let $G = (V, E)$ be a simple graph and let r denote the rank function of $\mathcal{C}(G)$. Then for each $E' \subseteq E$, we have*

$$r(E') = \min\{|F| + \sum_{X \in \mathcal{X}} (3|X| - 6) - \sum_{h \in H(\mathcal{X})} (\deg_{\mathcal{X}}(h) - 1)\},$$

where the minimum is taken over all subsets $F \subseteq E'$ and all 4-shellable 2-thin covers $\mathcal{X}$ of $(V, E' - F)$ with sets of size at least five.

Reconstruction from the cofactor matroid

Main difference from count matroid case: the number of vertices of G is **not** always determined by $\mathcal{C}_2^1(G)$, **even when the latter is connected!**

Reconstruction from the cofactor matroid

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Theorem (G-Jordán-Király 2024)

If G is 14-connected, then $\mathcal{C}_2^1(G)$ uniquely determines G.

Idea: we work with **higher (vertical) connectivity**.

Vertical connectivity

Definition

A matroid $\mathcal{M} = (E, r)$ is **connected** if for any bipartition $E = E_1 \cup E_2$ with $|E_1|, |E_2| \geq 1$, we have

$$r(E_1) + r(E_2) \geq r(E) + 1.$$

Vertical connectivity

Definition

A matroid $\mathcal{M} = (E, r)$ is k-**connected** if for any $k_0 \in \{1, \ldots, k-1\}$ and any bipartition $E = E_1 \cup E_2$ with $|E_1|, |E_2| \geq k_0$, we have

$$r(E_1) + r(E_2) \geq r(E) + k_0.$$

Vertical connectivity

Definition

A matroid $\mathcal{M} = (E, r)$ is **vertically k-connected** if for any $k_0 \in \{1, \dots, k-1\}$ and any bipartition $E = E_1 \cup E_2$ with $r(E_1), r(E_2) \geq k_0$, we have

$$r(E_1) + r(E_2) \geq r(E) + k_0.$$

Vertical connectivity

Definition

A matroid $\mathcal{M} = (E, r)$ is **vertically k-connected** if for any $k_0 \in \{1, \ldots, k-1\}$ and any bipartition $E = E_1 \cup E_2$ with $r(E_1), r(E_2) \geq k$, we have

$$r(E_1) + r(E_2) \geq r(E) + k_0.$$

"Vertical" comes from "vertex":

A graph G is k-connected

$$\Updownarrow$$

the graphic matroid $\mathcal{M}(G)$ is vertically k-connected.

Cofactor matroid facts for the proof

We show that in a 14-connected graph G we have

(complements of) vertex stars in G

$\Updownarrow$

vertically 8-connected 3-hyperplanes of $\mathcal{C}_2^1(G)$.

This follows from:

Theorem (G-Jordán-Király 2024)

- G is 13-connected $\Rightarrow \mathcal{C}_2^1(G)$ is vertically 8-connected.
- $\mathcal{C}_2^1(G)$ is vertically 8-connected $\Rightarrow \mathrm{rk}(\mathcal{C}_2^1(G)) = 3|V| - 6$.

$\rightarrow$ the number of vertices is determined by $\mathcal{C}_2^1(G)$!

The d-dimensional generic rigidity matroid

The generic rigidity matroid

For a graph G, the d-**dimensional generic rigidity matroid** $\mathcal{R}_d(G)$ of G is defined as the row matroid of a certain symbolic matrix.

We have

- $\mathcal{R}_1(G) =$ graphic matroid of G, and
- $\mathcal{R}_2(G) = (2,3)$-count matroid of G,

but for $d \geq 3$, no combinatorial characterization is known.

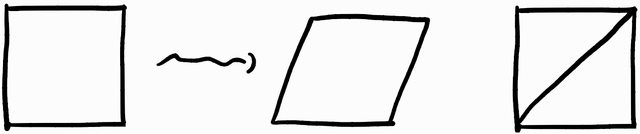
$\mathcal{R}_d$-rigid graphs

Definition

We say that a graph G on at least d vertices is **$\mathcal{R}_d$-rigid** if

$$\operatorname{rank}(\mathcal{R}_d(G)) = d|V(G)| - \binom{d+1}{2}.$$

Intuitively, this means that generic embeddings of G into $\mathbb{R}^d$ cannot be deformed continuously while keeping the edge lengths constant.



What do we know about the rigidity matroid?

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Main difference from cofactor case: here no combinatorial formula is known for the rank function (for $d \geq 3$)!

But! A recent breakthrough:

Theorem (Villányi 2025)

If G is $d(d+1)$-connected, then it is $\mathcal{R}_d$-rigid.

Connectivity and vertical connectivity

Theorem (Villányi 2025)

If G is $d(d+1)$-connected, then it is $\mathcal{R}_d$-rigid.

We can use this result (and very basic properties of $\mathcal{R}_d(G)$) to show that

Theorem (G 2024+)

- G is sufficiently highly connected $\Rightarrow \mathcal{R}_d(G)$ is highly vertically connected.
- $\mathcal{R}_d(G)$ is sufficiently highly vertically connected $\Rightarrow$ G is highly connected, and hence $\mathcal{R}_d$-rigid.

Identifying vertex stars in the rigidity matroid

This implies that for a $d(d+1)^2$-connected graph G,

(complements of) vertex stars in G

$\Updownarrow$

vertically $d^2(d+1)$-connected d-hyperplanes of $\mathcal{R}_d(G)$.

Theorem (G 2024+)

If G is $d(d+1)^2$-connected, then $\mathcal{R}_d(G)$ uniquely determines G.

Graph matroid families

A very general framework

Definition

A **graph matroid family (GMF)** $\mathcal{M}$ is a family of matroids $\mathcal{M}(G)$ defined on the edge set of each finite graph G satisfying the following properties:

- *well-defined*: isomorphic graphs get isomorphic matroids;
- *compatible*: taking subgraphs $\leftrightarrow$ restricting the matroid.

E.g., family $\mathcal{M}_{k,\ell}$ of (k,ℓ)-count matroids, family $\mathcal{R}_d$ of d-dimensional generic rigidity matroids...

Other names for (essentially) the same concept: 2-symmetric matroid (Kalai), matroidal family (Simões-Pereira), graph matroid limit (Király et. al)...

Dimensionality

A GMF $\mathcal{M}$ is **trivial** if $\mathcal{M}(G)$ is a free matroid for all G.

Lemma (G 2024+ / folklore?)

For every nontrivial GMF $\mathcal{M}$ there exist constants d, t, c such that for every $n \geq t$,

$$\mathrm{rank}(\mathcal{M}(K_n)) = dn - c.$$

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Definition

A graph G on $n \geq t$ vertices is **$\mathcal{M}$-rigid** if

$$\mathrm{rank}(\mathcal{M}(G)) = dn - c.$$

A general reconstructibility result

Theorem (G 2024+)

Let $\mathcal{M}$ be an unbounded GMF. Then

$\exists c$: every c-connected graph G is uniquely determined by $\mathcal{M}(G)$

$\Updownarrow$

$\exists c'$: every c'-connected graph is $\mathcal{M}$-rigid.

Proof: copy the proof for the generic rigidity matroid.

This recovers all previous "Whitney-type" results (sometimes with a worse bound on the required connectivity).

An open problem

The following is a strengthening of Whitney's theorem:

Theorem (Sanders, Sanders 1977)

Let G, H be graphs, and suppose that there is a bijection $f : E(G) \to E(H)$ such that for every circuit C in G, $f(C)$ is a circuit in H. If G is 3-connected, then G and H must be isomoprhic.

Is there a similar strengthening for other matroids / for graph matroid families?

References

Thank you!

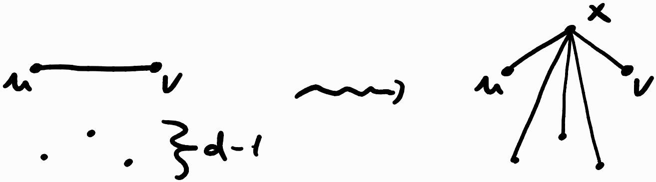
Some references:

- Garamvölgyi, **Rigidity and reconstruction in matroids of highly connected graphs**, 2024. *arXiv:2410.23431*
- Garamvölgyi, Jordán, Király, **Count and cofactor matroids of highly connected graphs**, *JCTB*, 2024.
- Jordán, Kaszanitzky, **Highly connected rigidity matroids have unique underlying graphs**, *EJC*, 2013.

Extendable graph matroid families

Given a graph G, the d-**dimensional edge split** operation replaces an edge uv of G with a new vertex joined to u and v, as well as to $d-1$ other vertices of G.

We say that a nontrivial graph matroid family $\mathcal{M}$ with dimensionality d is **extendable** if the d-dimensional edge split operation preserves independence in $\mathcal{M}(G)$, for all G.



The result of Villányi for GMFs

Theorem (Villányi 2025)

If G is $d(d+1)$-connected, then it is $\mathcal{R}_d$-rigid.

The same proof method can be used to show the following.

Theorem (Villányi 2025 / G 2024+)

If $\mathcal{M}$ is an extendable graph matroid family, then there exists a constant c such that every c-connected graph is $\mathcal{M}$-rigid.