Dissertation Defense
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Dániel Garamvölgyi

Unique reconstruction problems in rigidity theory

Supervisor: Tibor Jordán
Doctoral School: Mathematics

ELTE EÖTVÖS LORÁND UNIVERSITY
Alfréd Rényi Institute of Mathematics

The plan

- Quick introduction to combinatorial rigidity theory.
- Summary of main themes and results of the thesis.

Rigidity theory crash course

Rigid and globally rigid realizations

Let $G = (V, E)$ be a graph.

- A d-dimensional **realization** of G is a pair (G, p), where $p : V \to \mathbb{R}^d$.
- The **length** of an edge $uv \in E$ in (G, p) is $\|p(u) - p(v)\|$.

Rigid and globally rigid realizations

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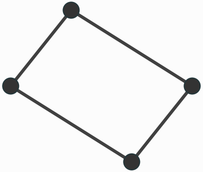
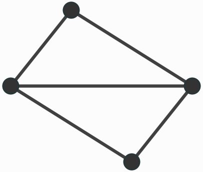
- A d-dimensional **realization** of G is a pair (G, p), where $p : V \to \mathbb{R}^d$.
- The **length** of an edge $uv \in E$ in (G, p) is $\|p(u) - p(v)\|$.
- (G, p) is **rigid** if it has no continuous edge length preserving deformations.
- (G, p) is **globally rigid** if the only d-dimensional realizations of G with the same edge lengths as (G, p) are the translations, rotations, and reflections of (G, p).

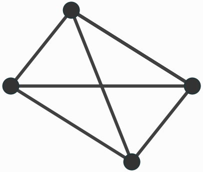
Examples



(a) Not rigid



(b) Not globally rigid



(c) Globally rigid

Graph rigidity

Theorem (Asimow-Roth + Gortler-Healy-Thurston)

- For any graph G, either all generic d-dimensional realizations of G are rigid, or none of them are.
- For any graph G, either all generic d-dimensional realizations of G are globally rigid, or none of them are.

Definition

A graph is **d-rigid** if its generic d-dimensional realizations are rigid.

A graph is **globally d-rigid** if its generic d-dimensional realizations are globally rigid.

*A realization is **generic** if the coordinates appearing in it form an algebraically independent set (over $\mathbb{Q}$).

The rigidity matroid

Let K_V be the complete graph on vertex set V.

The family

$$\mathcal{S} = \big\{E \subseteq E(K_V) \colon G = (V, E) \text{ is a } d\text{-rigid graph}\big\}$$

is the family of spanning sets of a matroid $\mathcal{R}_d(K_V)$.

Definition

The **(d-dimensional generic) rigidity matroid** of the graph $G = (V, E)$ is the restriction of $\mathcal{R}_d(K_V)$ to E. It is denoted by $\mathcal{R}_d(G)$.

Why study rigidity?

Rigidity theory has

- real-world applications (allegedly),
- applications to graph theory...
  (Motto: "d-rigidity is a matroidal version of d-connectivity")
- ...and to many other areas: circle packings, combinatorial geometry, low-rank matrix/tensor completion, etc.

Main themes of the thesis

The main themes/topics of the thesis are:

- combinatorial aspects of global rigidity,
- a theory of unlabeled reconstructibility,
- a toolbox for rigidity theory based on algebraic geometry.

Combinatorics of global rigidity

Rigidity vs global rigidity

Recall:

Theorem (Asimow-Roth 1978)

If any generic d-dimensional realization of G is rigid, then all of them are. (In this case, G is **d-rigid**.)

Theorem (Gortler-Healy-Thurston 2010)

If any generic d-dimensional realization of G is globally rigid, then all of them are. (In this case, G is **globally d-rigid**.)

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The Asimow-Roth result is **easy** and **robust**.

The Gortler-Healy-Thurston result is **hard** and (in a sense) a **fluke**.

Minimally rigid graphs

Definition

A graph $G = (V, E)$ is **minimally d-rigid** if it is d-rigid but $G - e$ is not, for every edge $e \in E$.

Minimally d-rigid graphs on vertex set V correspond to the bases of the rigidity matroid $\mathcal{R}_d(K_V)$ of the complete graph on V.

In particular, if $G = (V, E)$ is a minimally d-rigid graph with $|V| \geq d + 1$, then

- $|E| = d|V| - \binom{d+1}{2}$, and
- Any set $X \subseteq V$ of vertices with $|X| \geq d + 1$ induces at most $d|X| - \binom{d+1}{2}$ edges.

Minimally globally rigid graphs

Definition

A graph $G = (V, E)$ is **minimally globally d-rigid** if it is globally d-rigid but $G - e$ is not, for every edge $e \in E$.

Conjecture (Jordán 2017)

If $G = (V, E)$ is a minimally globally d-rigid graph with $|V| \geq d + 2$, then

$$|E| \leq (d+1)|V| - \binom{d+2}{2}.$$

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Equality holds if and only if $G = K_{d+2}$.

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Theorem (G 2024+)

If $G = (V, E)$ is a minimally globally d-rigid graph, then any set $X \subseteq V$ of vertices with $|X| \geq d + 2$ induces at most $(d+1)|X| - \binom{d+2}{2}$ edges.

The rigidity matroid of globally rigid graphs

Let G be a graph. Recall: $\mathcal{R}_d(G)$ denotes its rigidity matroid.

Observation

If G is globally d-rigid, then it is d-rigid.

($\Leftrightarrow \mathcal{R}_d(G)$ has maximum rank)

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If G is globally d-rigid, then $\mathcal{R}_d(G)$ is bridgeless.

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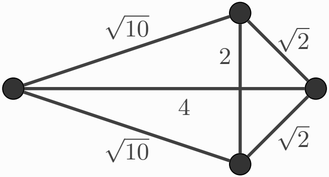
If G is globally d-rigid, then $\mathcal{R}_d(G)$ is bridgeless.

Theorem (G-Gortler-Jordán 2022)

If G is globally d-rigid, then $\mathcal{R}_d(G)$ is connected.

Unlabeled reconstruction

An interesting example



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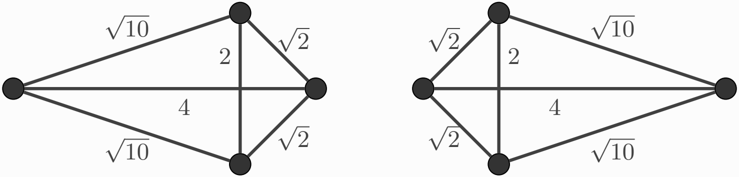

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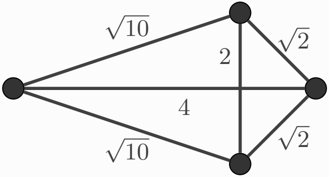

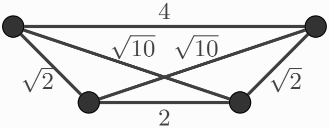

Comparing measurements

An **edge bijection** is a bijection $\psi : E(G) \to E(H)$ between the edge sets of two graphs.

Definition

Let (G, p) and (H, q) be realizations and let $\psi : E(G) \to E(H)$ be an edge bijection. We say that (G, p) and (H, q) are **length-equivalent** (under ψ) if for every edge $e \in E(G)$, the length of e in (G, p) is equal to the length of $\psi(e)$ in (H, q).

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An edge bijection $\psi : E(G) \to E(H)$ is **induced by a graph isomorphism** if it is of the form $\psi(uv) = \varphi(u)\varphi(v)$ for some graph isomorphism $\varphi : V(G) \to V(H)$.

Results on unlabeled reconstruction

Let K be the complete graph on n vertices, where $n \geq d + 2$.

Theorem (Boutin-Kemper 2004)

Let (K, p) and (K, q) be d-dimensional realizations that are length-equivalent under some edge bijection $\psi : E(K) \to E(K)$.
If (K, p) is generic, then ψ is induced by an automorphism of K.

Results on unlabeled reconstruction

Let G be a graph on n vertices, where $n \geq d + 2$.

Let H be another graph.

Theorem (Gortler-Theran-Thurston 2019)

Let (G, p) and (H, q) be d-dimensional realizations that are length-equivalent under some edge bijection $\psi : E(G) \to E(H)$.
If G is globally d-rigid, (G, p) is generic, and $|V(G)| = |V(H)|$, then ψ is induced by a graph isomorphism $\varphi : G \to H$.

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Algebraic geometry toolbox

The measurement map

Definition

The **measurement map** of G is the function $m_{d,G}$ mapping each d-dimensional realization (G, p) to

$$m_{d,G}(p) = \left(\|p(u) - p(v)\|^2\right)_{uv \in E}$$

This is a polynomial map $\rightarrow$ we can use algebraic geometry to study its properties.

The measurement variety

Definition

The d-dimensional **measurement variety** of G, denoted by $M_{d,G}$, is the smallest complex affine variety that contains the image of the measurement map $m_{d,G}$.

The geometry of $M_{d,G}$ encodes various rigidity-theoretic properties of G.

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The geometry of $M_{d,G}$ encodes various rigidity-theoretic properties of G. For example:

Lemma (G-Gortler-Jordán 2022)

The rigidity matroid $\mathcal{R}_d(G)$ is connected if and only if $M_{d,G}$ cannot be written as $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$ for any proper subgraphs G_1, G_2 of G.

Algebraic matroids

It turns out that the measurement map $m_{d,G}$ gives an **algebraic representation** of the rigidity matroid $\mathcal{R}_d(G)$:

A set of edges is independent in $\mathcal{R}_d(G)$

$\Updownarrow$

The corresponding coordinate functions of $m_{d,G}$ are algebraically independent

Algebraic matroids

It turns out that the measurement map $m_{d,G}$ gives an **algebraic representation** of the rigidity matroid $\mathcal{R}_d(G)$.

Similarly, the measurement variety can be seen as a **geometric representation** of the rigidity matroid:

$$\text{A set of edges is independent in } \mathcal{R}_d(G)$$
$$\Updownarrow$$
$$\text{The corresponding coordinate projection of } M_{d,G} \text{ is dominant}$$

("essentially surjective").

A bonus result

Theorem (G 2024+)

Let $A_1, \ldots, A_k \in \mathbb{R}^{m \times n}$, and let r be a positive integer. If $\operatorname{rank}(\sum_{i=1}^{k} A_i) \geq r$, then there is a subset $I \subseteq \{1, \ldots, k\}$ with $|I| \leq r$ such that $\operatorname{rank}(\sum_{i \in I} A_i) \geq r$.

This appears to be a new result. (Any applications?)

Literature for the dissertation

Core material:

- D. Garamvölgyi, S.J. Gortler, T. Jordán, Globally rigid graphs are fully reconstructible, *Forum of Mathematics, Sigma*, 2022.
- D. Garamvölgyi, T. Jordán, Graph reconstruction from unlabeled edge lengths, *Discrete & Computational Geometry*, 2021.
- D. Garamvölgyi, T. Jordán, Minimally globally rigid graphs, *European Journal of Combinatorics*, 2023.

Bits and pieces from:

- D. Garamvölgyi, Stress-linked pairs of vertices and the generic stress matroid, 2023. arXiv:2308.16851
- D. Garamvölgyi, T. Jordán, Global rigidity of unit ball graphs, *SIAM Journal on Discrete Mathematics*, 2020.
- D. Garamvölgyi, T. Jordán, Partial reflections and globally linked pairs in rigid graphs, *SIAM Journal on Discrete Mathematics*, 2024.