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Stress-linked pairs of vertices

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The plan

- Basic definitions: (global) rigidity,
  (globally) linked vertex pairs
- A stress-based sufficient condition for globally linked vertex pairs.

The basics

Standard definitions

Let $G = (V, E)$ be a graph.

- A d-dimensional **framework** is a pair (G, p), where $p : V \to \mathbb{R}^d$.

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- A d-framework is **generic** if ...
- Two d-frameworks (G, p) and (G, q) are **equivalent** if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\|, \quad \forall uv \in E.$$

$$\text{Eqv}(G, p) = \{q \in (\mathbb{R}^d)^V : (G, p) \text{ and } (G, q) \text{ are equivalent}\}$$

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- Two d-frameworks (G, p) and (G, q) are **congruent** if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\|, \quad \forall u, v \in V.$$

Rigidity

$$\text{Eqv}(G,p) = \{q \in (\mathbb{R}^d)^V : (G,p) \text{ and } (G,q) \text{ are equivalent}\}$$

Definition

A d-framework (G,p) is **rigid** if $\text{Eqv}(G,p)$ consists of finitely many congruence classes.

Definition

A graph G is **d-rigid** if every generic d-framework (G,p) is rigid.

Global rigidity

$$\mathrm{Eqv}(G,p) = \{q \in (\mathbb{R}^d)^V : (G,p) \text{ and } (G,q) \text{ are equivalent}\}$$

Definition

A d-framework (G,p) is **globally rigid** if $\mathrm{Eqv}(G,p)$ consists of a single congruence class.

Definition

A graph G is **globally d-rigid** if every generic d-framework (G,p) is globally rigid.

Rigidity is **hard** . . .

(Combinatorial characterization known for $d = 1, 2$, but long-standing open problem for $d \geq 3$.)

Rigidity vs. global rigidity 1

Rigidity is **hard** . . .

(Combinatorial characterization known for $d = 1, 2$, but long-standing open problem for $d \geq 3$.)

. . . and so is global rigidity.

(Combinatorial characterization known for $d = 1, 2$, but long-standing open problem for $d \geq 3$.)

Rigidity vs. global rigidity 2

But also, rigidity is **easy**!

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Rigidity vs. global rigidity 2

But also, rigidity is **easy**!

Matroid structure: if we can solve rigidity, we can automatically solve many related problems.

... global rigidity is still **hard**.

No matroid structure, many related algorithmic problems are NP-hard.

Linked vertex pairs

Definition

A vertex pair $\{u, v\}$ is **d-linked** in G if for every generic d-framework (G, p),

$$\{\, \|q(u) - q(v)\| : q \in \mathsf{Eqv}(G, p)\}$$

is finite.

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is finite.

- G is d-rigid $\Leftrightarrow$ every pair of vertices is d-linked in G.
- We can replace every with some and get the same notion.
- Matroid structure: deciding d-rigidity is polynomially equivalent to deciding d-linkedness.

Globally linked vertex pairs

Definition (Jackson, Jordán, Szabadka 2006)

A vertex pair $\{u, v\}$ is **globally d-linked** in G if for every generic d-framework (G, p),

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has size one.

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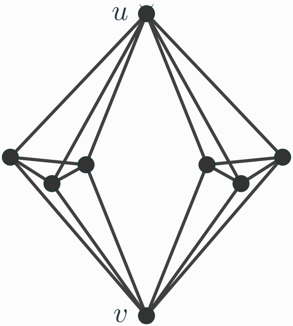
has size one.

- G is globally d-rigid $\Leftrightarrow$ every pair of vertices is globally d-linked in G.
- If we replace every with some, we get a different notion!
- Deciding global d-linkedness is open even in the $d = 2$ case!

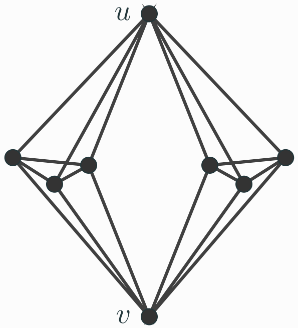
Short quiz

Quiz time: Double banana



Is $\{u, v\}$ 3-linked in B?

Quiz time: Double banana

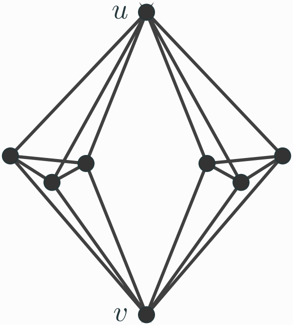


Is $\{u, v\}$ 3-linked in B?

Yes.

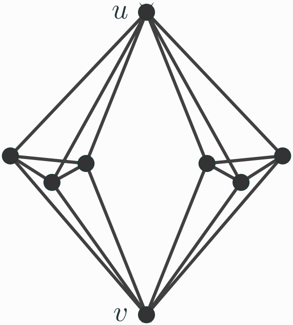
(It is contained in a 3-rigid subgraph.)

Quiz time: Double banana



Is $\{u, v\}$ globally 3-linked in B?

Quiz time: Double banana

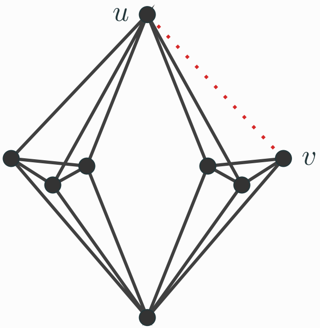


Is $\{u, v\}$ globally 3-linked in B?

Yes.

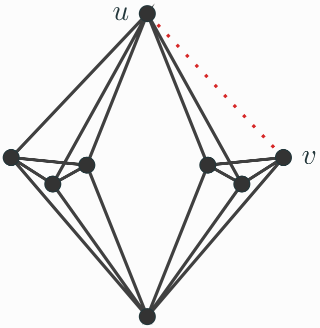
(Not entirely trivial!)

Quiz time: Double banana minus an edge



Is $\{u, v\}$ 3-linked in $B - uv$?

Quiz time: Double banana minus an edge

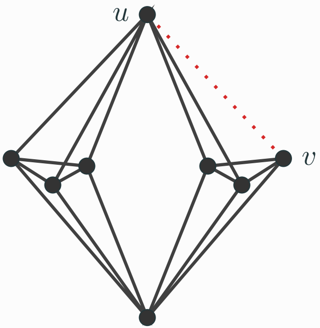


Is $\{u, v\}$ 3-linked in $B - uv$?

Yes.

(B is an "$\mathcal{R}_3$-circuit".)

Quiz time: Double banana minus an edge



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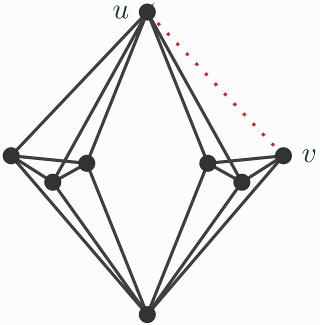
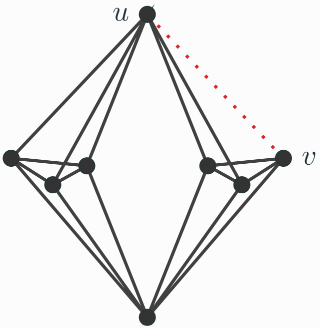


Is $\{u, v\}$ globally 3-linked in $B - uv$?

…probably not?

Quiz time: 2-sum of double bananas



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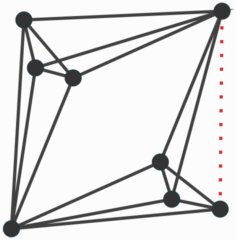

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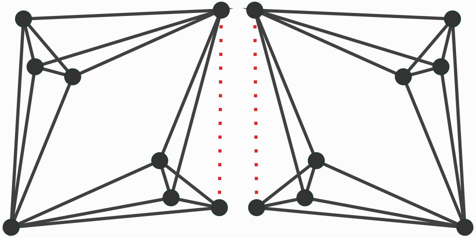

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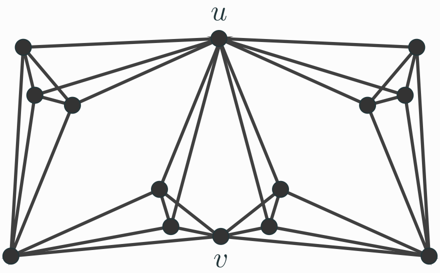

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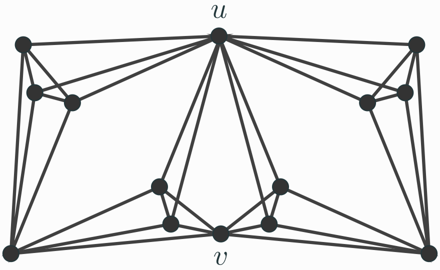
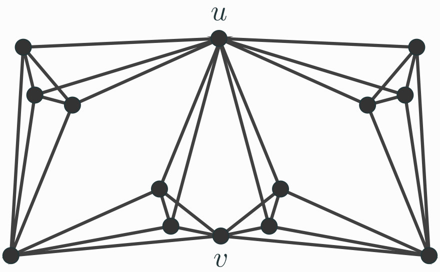


Is $\{u, v\}$ globally 3-linked in B'?

Quiz time: 2-sum of double bananas



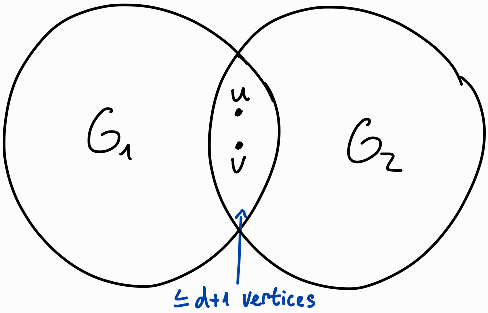
Is $\{u, v\}$ globally 3-linked in B'?

It is a secret. (Maybe we will find out . . .)

Gluing conjecture

Conjecture (G, Jordán 2024+)

Let G be the union of the graphs $G_1 = (V_1, E_1), i \in \{1, 2\}$ with $|V_1 \cap V_2| \leq d + 1$. Let $u, v \in V_1 \cap V_2$. If $\{u, v\}$ is d-linked in both G_1 and G_2, then $\{u, v\}$ is globally d-linked in G.



Stress-linked vertex pairs

Stresses of frameworks

Definition

A vector $\omega = (\omega_{uv})_{uv \in E}$ is a **stress** of (G, p) if it satisfies the following system of *equilibrium conditions*:

$$\sum_{u: uv \in E} \omega_{uv}(p(u) - p(v)) = 0, \qquad \forall v \in V.$$

The theorem of Connelly

Theorem (Connelly 2005 (and probably much earlier))

If (G, p) is generic and $q \in \mathsf{Eqv}(G, p)$, then every stress of (G, p) is also a stress of (G, q).

The theorem of Connelly

Theorem (Connelly 2005 (and probably much earlier))

If (G, p) is generic and $q \in \mathsf{Eqv}(G, p)$, then every stress of (G, p) is also a stress of (G, q).

Let us define

$$K(G,p) = \{q \in (\mathbb{R}^d)^V : \text{every stress of } (G,p) \text{ is a stress of } (G,q)\}$$

Then Connelly's theorem says:

If (G, p) is generic, then $\mathsf{Eqv}(G, p) \subseteq K(G, p)$.

The (other) Gortler-Healy-Thurston theorem

$$K(G,p) = \{q \in (\mathbb{R}^d)^V : \text{every stress of } (G,p) \text{ is a stress of } (G,q)\}$$

Easy observation:

$$K(G+uv,p) \subseteq K(G,p), \quad \forall u,v \in V.$$

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Theorem (Gortler, Healy, Thurston 2010)

A graph $G = (V,E)$ on at least $d+2$ vertices is globally d-rigid if and only if

$$K(G,p) = K(K_V,p)$$

for every generic d-framework (G,p).

Stress-linked vertex pairs: Definition

$$K(G,p) = \{q \in (\mathbb{R}^d)^V : \text{every stress of } (G,p) \text{ is a stress of } (G,q)\}$$

Definition (G 2024+)

A vertex pair $\{u,v\}$ is **d-stress-linked** in G if

- $\{u,v\}$ is d-linked in G, and
- $K(G+uv,p) = K(G,p)$ for every generic (G,p).

Stress-linked vs. globally linked

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We can replace every with some and get the same notion.

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Definition (Jackson, Jordán, Szabadka 2006)

A vertex pair $\{u, v\}$ is **globally d-linked** in G if for every generic d-framework (G, p),

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has size one.

Stress-linked vertex pairs: Main theorem

Theorem (G 2024+)

If $\{u, v\}$ is d-stress-linked in G, then it is globally d-linked in G.

Proof idea:

- *Linearity of the generic contact locus:*
  The image of $K(G + uv, p)$ under the mapping
  $$q \mapsto \left(\left\|q(u') - q(v')\right\|^2\right)_{u'v' \in E(G+uv)}$$
  is "almost" a linear space.
- If $\{u, v\}$ is not globally d-linked in G, then we can use the previous point, Connelly's theorem, and the assumption that $K(G, p) = K(G + uv, p)$ to contradict d-linkedness.

Stress-linked vertex pairs: Gluing theorem

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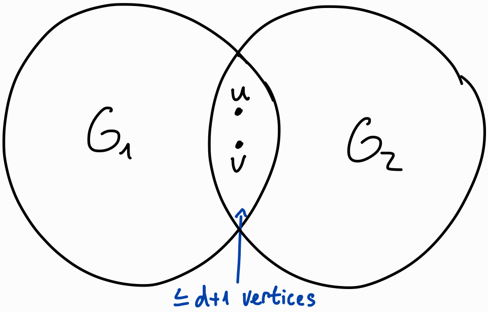

Stress-linked vertex pairs: Gluing theorem

Theorem (G 2024+)

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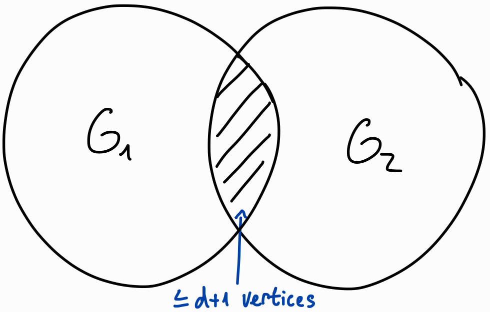
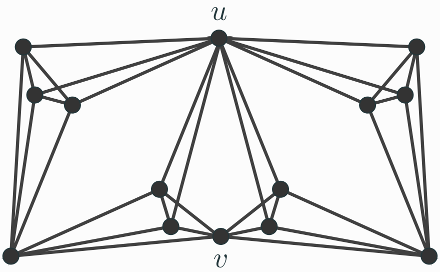


2-sum of double bananas, revisited



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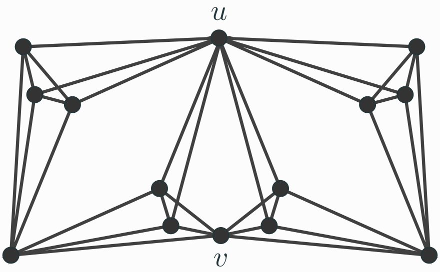

Is $\{u, v\}$ globally 3-linked in B'?

Yes!

3-linked on both sides $\Rightarrow$ 3-stress-linked in B' $\Rightarrow$ globally 3-linked in B'

Applications

Some other applications of stress-linked vertex pairs:

- Sparsity result for minimally globally d-rigid graphs.
- A gluing result for redundantly d-rigid graph (answering a conjecture of Connelly).
- A gluing result for $\mathcal{R}_d$-circuits (answering a conjecture of Grasegger, Guler, Jackson and Nixon).

Stress-linked vertex pairs: Tools

Main tools for studying stress-linked vertex pairs:

- Extension of rigidity theory to complex space.
- Duality theory of projective varieties.
- Viewpoint of algebraic matroids.

The big question

Conjecture (G 2024+)

A pair of vertices $\{u, v\}$ is globally d-linked in G if and only if it is d-stress-linked in G.

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Preprint: D. Garamvölgyi, Stress-linked pairs of vertices and the generic stress matroid, 2023. arXiv:2308.16851

See also: D. Garamvölgyi, T. Jordán, Partial reflections and globally linked pairs in rigid graphs, should appear soon in *SIDMA*.