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# Rigidity and reconstruction in matroids of highly connected graphs

Partly based on joint work with Tibor Jordán, Csaba Király, and Soma Villányi

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# This talk

Let $\mathcal{M}(G)$ be a matroid defined on the edge set of the graph $G$, for each finite graph $G$. We will consider two types of questions:

- **Whitney type**: Is it true that every sufficiently highly connected graph $G$ is uniquely determined by $\mathcal{M}(G)$?
- **Lovász-Yemini type**: Is it true that for every sufficiently highly connected graph $G$, $\mathcal{M}(G)$ is of maximum rank among graphs on the same number of vertices?

These questions turn out to be closely related.

# Basic examples: Classic theorems

Let $\mathcal{M}_{1,1}(G)$ denote the graphic matroid of $G$.

**Theorem (Whitney 1933)**

Let $G$ and $H$ be graphs, and let $\psi : E(G) \to E(H)$ be an isomorphism between $\mathcal{M}_{1,1}(G)$ and $\mathcal{M}_{1,1}(H)$. If $G$ is $3$-connected and $H$ is without isolated vertices, then it $\psi$ is induced by a graph isomorphism.

Let $\mathcal{R}_2(G)$ denote the $2$-dimensional generic rigidity matroid of $G$.

**Theorem (Lovász and Yemini 1982)**

If $G$ is $6$-connected, then the rank of $\mathcal{R}_2(G)$ is maximal among graphs on $|V(G)|$ vertices.

# Setting

# The general setting: Graph matroid families

A **graph matroid family** $\mathcal{M}$ is a family of matroids $\mathcal{M}(G)$, defined on the edge set of each finite graph $G$, that is

- *well-defined*: every graph isomorphism $\varphi : V(G) \to V(H)$ induces an isomorphism between $\mathcal{M}(G)$ and $\mathcal{M}(H)$;
- *compatible*: if $H$ is a subgraph of $G$, then the restriction of $\mathcal{M}(G)$ to $E(H)$ is $\mathcal{M}(H)$.

Equivalently, $\mathcal{M}$ is a finitary matroid on the edge set of the countable complete graph $K_{\mathbb{N}}$ that is invariant under the automorphisms of $K_{\mathbb{N}}$.

Other names for (essentially) the same concept: $2$-symmetric matroid (Kalai), matroidal family (Simões-Pereira), graph matroid limit (Király et. al)...

# Basic properties of graph matroid families

Let $\mathcal{M}$ be a graph matroid family, and let $r(G)$ denote the rank of $\mathcal{M}(G)$, for each graph $G$. A graph $G$ is

- $\mathcal{M}$-**independent** if $\mathcal{M}(G)$ is a free matroid;
- $\mathcal{M}$-**rigid** if $r(G) = r(K_V)$, where $K_V$ is the complete graph on the vertex set of $G$.

The graph matroid family $\mathcal{M}$ is **trivial** if every graph is $\mathcal{M}$-independent, and **nontrivial** otherwise.

Not difficult to show: if $\mathcal{M}$ is nontrivial, then there exist integers $d$ and $c$ such that for large enough $n$, $r(K_n) = dn + c$. Let us call this $d$ the **dimensionality** of $\mathcal{M}$.

# Examples 1: Count matroids

Let $k, \ell$ be integers with $k \geq 1$ and $\ell \leq 2k - 1$.

A graph is $(k, \ell)$-**sparse** if every subset $V'$ of vertices of size at least two induces at most $k|V'| - \ell$ edges.

The family of $(k, \ell)$-**count matroids** is the graph matroid family $\mathcal{M}_{k,\ell}$ defined by

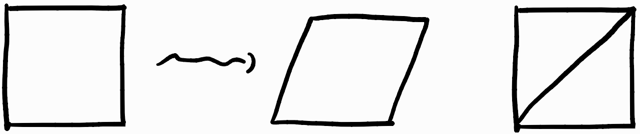
$$G \text{ is } \mathcal{M}_{k,\ell}\text{-independent} \iff G \text{ is } (k, \ell)\text{-sparse.}$$

- $\mathcal{M}_{1,1}(G)$ = graphic matroid of $G$,
- $\mathcal{M}_{k,k}(G)$ = $k$-fold union of graphic matroid of $G$,
- $\mathcal{M}_{1,0}(G)$ = bicircular matroid of $G$.

## Examples 2: The generic rigidity matroid

Fix $d \geq 1$. For every graph $G$, the $d$-**dimensional generic rigidity matroid** of $G$, denoted by $\mathcal{R}_d(G)$, is defined as the row matroid of a certain symbolic matrix.

Intuitively, a graph is $\mathcal{R}_d$-rigid if its generic embeddings into $\mathbb{R}^d$ cannot be deformed continuosly while keeping the edge lengths constant.



# Properties of the generic rigidity matroid

Key property for us:

$\mathcal{R}_d$-rigid graphs on at least $d+1$ vertices are $d$-connected.

We have

- $\mathcal{R}_1(G) = \mathcal{M}_{1,1}(G)$ (easy), and
- $\mathcal{R}_2(G) = \mathcal{M}_{2,3}(G)$ (a theorem of Laman and Pollaczek-Geiringer).

For $d \geq 3$, $\mathcal{R}_d(G)$ is not a count matroid, and finding a good characterization for its rank function is a major open problem.

# Lovász-Yemini type results

# LY type results for count matroids 1

By a **Lovász-Yemini type result**, I mean a statement of the form "every $c$-connected graph is $\mathcal{M}$-rigid."

For count matroids, we have the following basic results.

- By definition, every connected graph is $\mathcal{M}_{1,1}$-rigid.
- Every $2$-connected graph is $\mathcal{M}_{1,0}$-rigid. In fact, minimum degree at least $2$ suffices.
- Every $2k$-connected graph is $\mathcal{M}_{k,k}$-rigid. In fact, $2k$-edge-connectivity suffices.
- By the theorem of Lovász and Yemini (and Laman and Pollaczek-Geiringer), every $6$-connected graph is $\mathcal{M}_{2,3}$-rigid.

All of these constants are best possible.

# LY type results for count matroids 2

**Theorem (G, Jordán, Király 2024)**

Every $c$-connected graph is $\mathcal{M}_{k,\ell}$-rigid, where

- $c = \lceil 2k - 2\ell/|V(G)| \rceil$ if $\ell \leq 0$,
- $c = \max(2k, 2\ell)$ if $\ell \geq 0$.

Moreover,

- if $\ell \leq 0$, then minimum degree at least $2k - 2\ell/|V(G)|$ suffices for $\mathcal{M}_{k,\ell}$-rigidity;
- if $0 \leq \ell \leq k$, then $2k$-edge-connectivity suffices for $\mathcal{M}_{k,\ell}$-rigidity.

The difficult part is the $k < \ell \leq 2k - 1$ case. The proof is a tricky adaptation of the original Lovász-Yemini proof.

# LY type results for generic rigidity matroids

Lovász and Yemini also conjectured an extension of their theorem for $\mathcal{R}_d(G), d \geq 3$.

**Conjecture (Lovász and Yemini 1982)**

Every $d(d+1)$-connected graph is $\mathcal{R}_d$-rigid.

They gave an example showing that $d(d+1)$ would be best possible.

Recently, Villányi gave an amazing proof for this conjecture.

**Theorem (Villányi 2024+)**

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# LY type results for unions of generic rigidity matroids

Let $\mathcal{R}_d^k$ denote the $k$-fold union of $\mathcal{R}_d$.

The proof of Villányi can be adapted show a Lovász-Yemini type result for $\mathcal{R}_d^k$ with constant of order $O(k^2d^2)$.

Using another proof method, we could improve the dependence on $k$.

**Theorem (G, Jordán, Király, Villányi 2024+)**

Every $(k \cdot 10d(d+1))$-connected graph is $\mathcal{R}_d^k$-rigid.

This is not best possible; maybe the optimal bound is $k \cdot d(d+1)$.

# Applications 1: A conjecture of Kriesell

Observation: an $\mathcal{R}_d^k$-rigid graph on at least $2kd$ vertices contains $k$ edge-disjoint $\mathcal{R}_d$-rigid (and in particular, $d$-connected) spanning subgraphs. Hence we have the following corollary.

**Corollary (G, Jordán, Király, Villányi 2024+)**

Every $(k \cdot 10d(d+1))$-connected graph contains $k$ edge-disjoint $d$-connected spanning subgraphs.

In particular, this confirms the following conjecture of Kriesell.

**Conjecture (Kriesell, ∼2003)**

For every positive integer $d$, there exists a (smallest) integer $g(d)$ such that every $g(d)$-connected graph $G$ contains a spanning tree $T$ for which $G - E(T)$ is $d$-connected.

# Applications 2: A conjecture of Thomassen

A directed graph is $k$-**connected** if it has at least $k+1$ vertices and it remains strongly connected after the removal of any set of less than $k$ vertices.

**Theorem (G, Jordán, Király, Villányi 2024+)**

For every positive integer $k$, there exists a (smallest) integer $f(k)$ such that every $f(k)$-connected graph has a $k$-connected orientation.

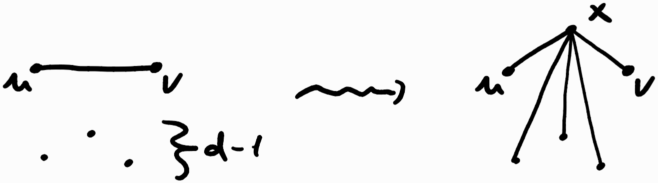
This confirms a conjecture of Thomassen from the '80s.

To prove this, we show that if a graph contains two edge-disjoint $\mathcal{R}_{4k}$-rigid spanning subgraphs, then it has a $k$-connected orientation. The bound we get is $f(k) = O(k^2)$, but the constant factor is rather large.

# Extendable graph matroid families

Given a graph $G$, the $d$-**dimensional edge split** operation replaces an edge $uv$ of $G$ with a new vertex joined to $u$ and $v$, as well as to $d-1$ other vertices of $G$.

We say that a nontrivial graph matroid family $\mathcal{M}$ with dimensionality $d$ is **extendable** if the $d$-dimensional edge split operation preserves $\mathcal{M}$-independence.



# A general Lovász-Yemini type result

Let us say that a graph matroid family $\mathcal{M}$ has the **Lovász-Yemini property** if there is a constant $c$ such that every $c$-connected graph is $\mathcal{M}$-rigid.

The proof method of Villányi can be used to show the following.

**Theorem (G 2024++)**

If a graph matroid family is extendable, then it has the Lovász-Yemini property.

This implies all of the Lovász-Yemini type results we have seen so far (but the constant $c$ we get can be far from best possible).

# Whitney type results

# Reconstructibility from a matroid

Let $\mathcal{M}$ be a graph matroid family. We say that a graph $G$ without isolated vertices is $\mathcal{M}$-**reconstructible** if $\mathcal{M}(G)$ uniquely determines $G$, in the following sense: if $H$ is another graph without isolated vertices and $\psi : E(G) \to E(H)$ is an isomorphism between $\mathcal{M}(G)$ and $\mathcal{M}(H)$, then $\psi$ is induced* by a graph isomorphism.

(A graph isomorphism $\varphi : V(G) \to V(H)$ **induces** $\psi$ if $\psi(uv) = \varphi(u)\varphi(v)$ holds for every edge $uv \in E(G)$.)

# Whitney type results for count matroids 1

By a **Whitney type result**, I mean a statement of the form "every $c$-connected graph is $\mathcal{M}$-reconstructible."

- Whitney characterized isomorphisms of graphic matroids. His theorem implies that $3$-connected graphs are $\mathcal{M}_{1,1}$-reconstructible.
- A theorem of Wagner implies that $3$-connected graphs are $\mathcal{M}_{1,0}$-reconstructible.
- Jordán and Kaszanitzky showed that $7$-connected graphs are $\mathcal{M}_{2,3}$-reconstructible.

The first two results are best possible. For the third, it is open whether $6$-connectivity suffices.

# Whitney type results for count matroids 2

We have the following in the general case.

**Theorem (G, Jordán, Király 2024)**

Every $c$-connected graph is $\mathcal{M}_{k,\ell}$-reconstructible, where

- $c = \lceil 2k - (2\ell - 2)/|V(G)| \rceil$ if $\ell \leq 0$
- $c = \max(2k, 2\ell) + 1$ if $\ell \geq 0$.

These bounds are almost tight (off by at most one).

The proof uses a strengthening of the corresponding Lovász-Yemini type results, and the fact that the circuits of $\mathcal{M}_{k,\ell}(G)$ induce $\mathcal{M}_{k,\ell}$-rigid subgraphs.

# A general Whitney type result

A graph matroid family $\mathcal{M}$ has the **Whitney property** if there is a constant $c$ such that every $c$-connected graph is $\mathcal{M}$-reconstuctible.

The family $\mathcal{M}$ is **unbounded** if the rank of $\mathcal{M}(K_n)$ tends to infinity as $n \to \infty$.

**Theorem (G 2024++)**

An unbounded graph matroid family has the Lovász-Yemini property if and only if it has the Whitney property.

In the bounded case, the Lovász-Yemini property always holds, and families with the Whitney property have a reasonably good characterization.

# Future directions

Short recap:

- Lovász-Yemini and Whitney type results are closely related and hold for many graph matroid families.
- Lovász-Yemini type theorems can have interesting graph-theoretic consequences.

Possible future directions (among many):

- Is there a graph matroid family that is not extendable but has the Lovász-Yemini property?
- Other settings: matroids on hypergraphs, bipartite graphs, group-labeled graphs...
- Algorithms for reconstructing the underlying graph from the matroid (seems to be open except for graphic matroids).

# Papers

[1] D. Garamvölgyi., Rigidity and reconstruction in matroids of highly connected graphs, *In preparation.*

[2] D. Garamvölgyi, T. Jordán, Cs. Király, Count and cofactor matroids of highly connected graphs, *Journal of Combinatorial Theory, Series B*, 2024.

[3] D. Garamvölgyi, T. Jordán, Cs. Király, S. Villányi, Highly connected orientations from edge-disjoint rigid subgraphs, arXiv:2401.12670, 2024.

[4] S. Villányi, Every $d(d+1)$-connected graph is globally rigid in $\mathbb{R}^d$, arXiv:2312.02028, 2023.