# Partial reflections and globally linked pairs

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2023. August 10.

# This talk

In this talk I will

- give the basics of the combinatorics of globally linked vertex pairs,
- describe some graph families in which this notion is well-understood,
- explain a connection between a combinatorial condition on globally linked pairs and the geometric construction of taking (sequences of) “partial reflections”.

# Definitions – frameworks

Let us fix $d \geq 1$ and a graph $G$. A **framework**, or a **realization** of $G$ is a pair $(G, p)$ with $p : V(G) \to \mathbb{R}^d$.

Two realizations $(G, p)$ and $(G, q)$ are **equivalent** if

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \quad \forall uv \in E(G).$$

They are **congruent** if

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \quad \forall u, v \in V(G).$$

A realization $(G, p)$ is **generic** if the coordinates of $p(v), v \in V(G)$ are algebraically independent over $\mathbb{Q}$.

# Definitions – globally linked pairs in frameworks

Let $u, v \in V(G)$ be a pair of vertices and $(G, p)$ a framework. We say that $\{u, v\}$ is **globally linked in** $(G, p)$ if for every equivalent framework $(G, q)$ we have

$$||p(u) - p(v)|| = ||q(u) - q(v)||.$$

In other words, $(G + uv, p)$ and $(G + uv, q)$ are also equivalent.

if every pair of vertices is globally linked in $(G, p)$, then $(G, p)$ is **globally rigid**. This means that every framework equivalent to $(G, p)$ is congruent to $(G, p)$.

# Demotivating example 1

"Being globally linked" is not a generic property!

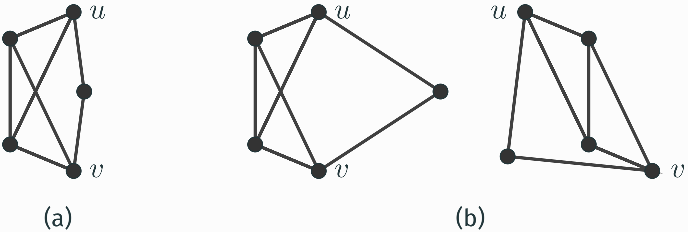


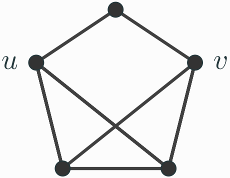

**Figure 1:** The pair $\{u, v\}$ is globally linked in the realization depicted in *(a)*, but not in the realizations depicted in *(b)*.

On the other hand, "being globally rigid" **is** a generic property.

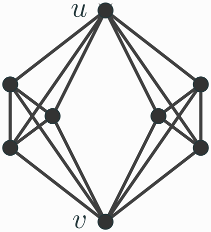
# Definitions – globally linked pairs in graphs

Let $u, v \in V(G)$ be a pair of vertices. We say that $\{u, v\}$ is **globally linked in $G$ in $\mathbb{R}^d$** if $\{u, v\}$ is globally linked in every generic $d$-dimensional realization of $G$.

The graph $G$ is **globally rigid in $\mathbb{R}^d$** if every generic $d$-dimensional realization of $G$ is globally rigid.




(a)




(b)

## Basic observations

- If $uv \in E(G)$, then $\{u, v\}$ is (trivially) globally linked in $G$ in $\mathbb{R}^d$.

Let $\kappa_G(u, v)$ denote the minimum size of a separator in $G$ that disconnects $u$ and $v$.

- If $u$ and $v$ are nonadjacent and globally linked in $\mathbb{R}^d$, then $\kappa_G(u, v) \geq d + 1$.

  (Indeed, if there is a small separator that disconnects $u$ and $v$, then we can perform a **partial reflection** using this separator.)
- For $d = 1$, this is sufficient: $\{u, v\}$ is globally linked in $\mathbb{R}^1$ if and only if $uv \in E(G)$ or $\kappa_G(u, v) \geq 2$.

# Partial reflections

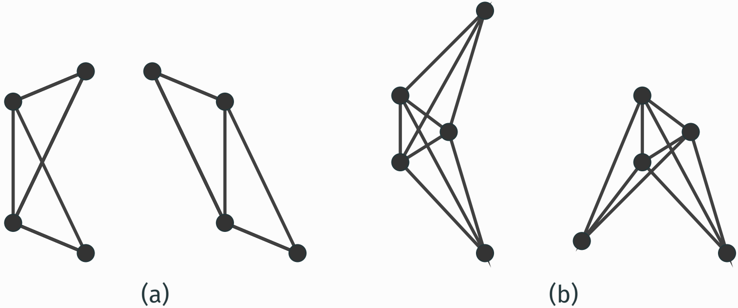



Figure 3: Partial reflections *(a)* in two dimensions, and *(b)* in three dimensions.

# The $\mathcal{R}_2$-connected theorem

**Theorem.** (Jackson, Jordán, Szabadka, 2006)
If $G$ is $\mathcal{R}_2$-connected, then we have the following.

*(a)* A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^2$ $\iff uv \in E(G)$ or $\kappa_G(u, v) \geq 3$.

*(b)* In particular, "being globally linked" is a generic property of $2$-dimensional realizations of $G$.

*(c)* For every generic $2$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

## The braced MOP theorem

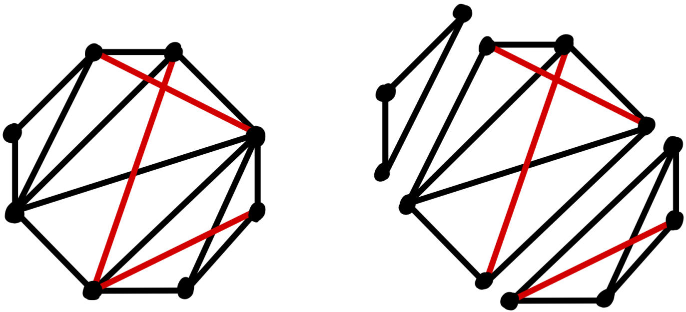
**Theorem.** (G., Jordán, 2022)
If $G$ is a braced maximal outerplanar graph, then we have the following.

(*a*) A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^2$
$\iff uv \in E(G)$ or $\kappa_G(u, v) \geq 3$.

(*b*) In particular, "being globally linked" is a generic property of $2$-dimensional realizations of $G$.

(*c*) For every generic $2$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

# A braced MOP



In a braced maximal outerplanar graph, each bracing edge actually creates a globally rigid subgraph.

# Questions

(*a*) A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^2$
$\iff uv \in E(G)$ or $\kappa_G(u, v) \geq 3$.

(*b*) In particular, "being globally linked" is a generic property of $2$-dimensional realizations of $G$.

(*c*) For every generic $2$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

Questions:

- For which graphs $G$ does *(a)* (and hence *(b)*) hold?
- What is the precise relationship between *(a)* and *(c)*?

# Demotivating example 2

Consider the cycle $C_4$ in the plane. It satisfies property *(a)*, but not property *(c)*.

The problem is that the two separating vertex pairs are *crossing*.

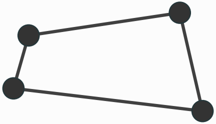


Figure 4: $C_4$ realized generically in the plane.

# Evolution of the braced MOP theorem

**Theorem.** (G., Jordán, 2022)
If $G$ is a braced maximal outerplanar graph, then we have the following.

*(a)* A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^2$ $\iff uv \in E(G)$ or $\kappa_G(u, v) \geq 3$.

*(b)* In particular, "being globally linked" is a generic property of $2$-dimensional realizations of $G$.

*(c)* For every generic $2$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

# Evolution of the braced MOP theorem

**Theorem.** (G., Jordán, 2023+)
If $G$ is a braced $2$-tree, then we have the following.

(*a*) A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^2$ $\iff uv \in E(G)$ or $\kappa_G(u, v) \geq 3$.

(*b*) In particular, "being globally linked" is a generic property of $2$-dimensional realizations of $G$.

(*c*) For every generic $2$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

($2$-tree $\Leftrightarrow$ obtained by gluing triangles along edges)

# Evolution of the braced MOP theorem

**Theorem.** (G., Jordán, 2023+)
If $G$ is a braced $d$-tree, then we have the following.

(*a*) A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^d$
$\iff uv \in E(G)$ or $\kappa_G(u, v) \geq d + 1$.

(*b*) In particular, "being globally linked" is a generic property of $d$-dimensional realizations of $G$.

(*c*) For every generic $d$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

($d$-tree $\Leftrightarrow$ obtained by gluing complete graphs of size $d + 1$ along complete graphs of size $d$)

# Evolution of the braced MOP theorem

**Theorem.** (G., Jordán, 2023+)

If $G$ is a braced $d$-connected chordal graph, then we have the following.

(*a*) A pair of vertices $\{u, v\}$ is globally linked in $G$ in $\mathbb{R}^d$ $\iff uv \in E(G)$ or $\kappa_G(u, v) \geq d + 1$.

(*b*) In particular, “being globally linked” is a generic property of $d$-dimensional realizations of $G$.

(*c*) For every generic $d$-dimensional realization $(G, p)$, every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

($d$-connected chordal graph $\Leftrightarrow$ obtained by gluing complete graphs of size at least $d + 1$ along complete graphs of size at least $d$)

# Changing focus: $d$-joined graphs

Let us say that a graph $G$ is $d$-**joined** if it is rigid in $\mathbb{R}^d$ and it satisfies property *(a)* above, that is,

$$\{u, v\} \text{ is globally linked in } G \text{ in } \mathbb{R}^d$$
$$\Updownarrow$$
$$uv \in E(G) \text{ or } \kappa_G(u, v) \geq d + 1.$$

By the previous results, $\mathcal{R}_2$-connected graphs and (braced) MOPs are $2$-joined, and (braced) $d$-connected chordal graphs are $d$-joined.

# Main results on $d$-joined graphs

**Theorem.** *Geometric characterization.* (G., Jordán, 2023+)
If $G$ is $d$-joined and $(G, p)$ is a generic realization in $\mathbb{R}^d$, then every equivalent realization can be obtained by a sequence of partial reflections (up to congruence).

**Theorem.** *Monotonicity.* (G., Jordán, 2023+)
If $G$ is $d$-joined then is $G + uv$ also $d$-joined, for any $u, v \in V(G)$.

**Theorem.** *Gluing theorem.* (G., Jordán, 2023+)
If $G_1, G_2$ are $d$-joined and $|V(G_1) \cap V(G_2)| \geq d$, then $G_1 \cup G_2$ is also $d$-joined.

# Proof strategy

To prove the geometric characterization, we show that there are no crossing $d$-separators in rigid graphs.

To prove monotonicity and the gluing theorem:

- We work with the geometric characterization.
- We have to understand the action of “sequences of partial reflections” on generic frameworks and relate it to edge addition and gluing.

Main difficulty: “sequences of partial reflections” are clunky!

# Globally rigid gluing constructions

**Theorem.** *Gluing theorem.* (G., Jordán, 2023+)
If $G_1, G_2$ are $d$-joined and $|V(G_1) \cap V(G_2)| \geq d$, then $G_1 \cup G_2$ is also $d$-joined.

A graph $G$ has a **globally rigid gluing construction in** $\mathbb{R}^d$ if there is a sequence of graphs $G_1, \ldots, G_l = G$ such that each $G_i$ is either globally rigid in $\mathbb{R}^d$, or is obtained by gluing some earlier graphs $G_j$ and $G_k$ on at least $d$ vertices.

**Corollary.** (G., Jordán, 2023+)
If $G$ has a globally rigid gluing construction in $\mathbb{R}^d$, then it is $d$-joined. In particular, such a graph is globally rigid in $\mathbb{R}^d$ if and only if it is $(d+1)$-connected.

# Future problems

- Big goal: give a good (not necessarily combinatorial) characterization of globally linked pairs in $\mathbb{R}^d$.
- (Probably) more manageable goal: give a combinatorial characterization of $2$-joined graphs.
- Is there a "nice" extension of the notion of $d$-joined graphs that also includes braced triangulations (for $d = 3$)?

**Theorem.** (Jordán, Tanigawa, 2019)
If $G$ is a braced triangulation and $u, v \in V(G)$, then $\{u, v\}$ is globally linked in $\mathbb{R}^3 \iff uv \in E(G)$ or $\kappa_G(u, v) \geq 4$ and the "$4$-block" of $\{u, v\}$ contains at least one bracing edge.

# Papers

[1] D.G., T. Jordán, “Globally linked pairs in braced maximal outerplanar graphs”, *Proceedings of the 34th Canadian Conference on Computational Geometry*, 2022.

[2] D.G., T. Jordán, “Partial reflections and globally linked pairs in rigid graphs”, *arXiv preprint*, 2023.