Minimally globally rigid graphs

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This talk

Conjecture. (Jordán, $\sim$2015)
If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

This talk

Theorem. (G., Jordán, 2022)
If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

In this talk:

- What does this mean?
- Why do we care?
- How did we prove it?
- What is (I think) the "real" reason that it is true?
- Some very recent results.

Definitions – realizations

What does "minimally globally rigid" mean?

Let us fix $d \geq 1$ and a graph G. A **framework**, or a **realization** of G is a pair (G, p) with $p : V(G) \to \mathbb{R}^d$. Two realizations (G, p) and (G, q) are **equivalent** if

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \quad \forall uv \in E(G).$$

They are **congruent** if

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \quad \forall u, v \in V(G).$$

A realization (G, p) is **generic** if the coordinates of $p(v), v \in V(G)$ are algebraically independent over $\mathbb{Q}$.

Definitions – rigidity

The framework (G, p) is **rigid** if it cannot be deformed continuously and **globally rigid** if every equivalent realization (G, q) is congruent to (G, p).

The graph G is **rigid in** $\mathbb{R}^d$ if its generic realizations in $\mathbb{R}^d$ are rigid. It is **globally rigid in** $\mathbb{R}^d$ if its generic realizations in $\mathbb{R}^d$ are globally rigid.

We say that G is **minimally globally rigid in** $\mathbb{R}^d$ if it is globally rigid, but $G - e$ is not globally rigid for any edge $e \in E(G)$. We define minimally rigid graphs similarly.

Matroid structure

The (edge sets of) minimally rigid graphs on n vertices form the bases of a matroid. In particular, they always have the same number of edges. This number is $r_d(K_n)$, the rank of the generic rigidity matroid in d dimensions and on n vertices.

The (edge sets of) minimally globally rigid graphs do not form a matroid.

For example, if $d = 1$, then

- minimally rigid graphs $\Leftrightarrow$ trees,
- minimally globally rigid graphs $\Leftrightarrow$ minimally 2-connected graphs.

Back to the theorem – why do we care?

Theorem.

If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

Why do we care?

- It's a nice clean result.
- Studying the minimal elements of a graph family may be useful for combinatorial characterizations.
- In particular, by simple counting we have the

Corollary. (G., Jordán, 2022)

If G is a minimally globally rigid graph in $\mathbb{R}^d$, then G has a vertex of degree at most $2d+1$.

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Back to the theorem – where does the bound come from?

Theorem.
If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

What does $n(d+1) - \binom{d+2}{2}$ mean?

The meaning of the bound

First idea:

$$n(d+1) - \binom{d+2}{2} = r_{d+1}(K_n),$$

i.e., the number of edges required for rigidity in $d+1$ dimensions.

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Second idea:

$$\begin{aligned} n(d+1) - \binom{d+2}{2} &= nd - \binom{d+1}{2} + n - d - 1 \\ &= r_d(K_n) + n - d - 1 \end{aligned}$$

… what does $n - d - 1$ mean?

What does $n-d-1$ mean?

Let (G,p) be a realization in $\mathbb{R}^d$. A symmetric matrix $\Omega \in \mathbb{R}^{n\times n}$ is a **stress matrix** of (G,p) if

- it is a "G-matrix": $\Omega(ij)=0$ whenever $v_iv_j \notin E(G)$, and
- equilibrium condition: $P\Omega = 0$, where

$$P = \begin{pmatrix} p(v_1) & p(v_2) & \dots & p(v_n) \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times n}$$

Theorem. (Connelly, 2005 + Gortler, Healy, Thurston, 2010)
The graph G on $n \geq d+2$ vertices is globally rigid in $\mathbb{R}^d$
$\Leftrightarrow$ some generic (G,p) has a stress matix of rank $n-d-1$
$\Leftrightarrow$ every generic (G,p) has a stress matix of rank $n-d-1$.

Fundamental stresses

Given a generic realization (G, p), the stress matrices of (G, p) form a linear space. We can describe a basis of this space as follows:

- Fix any maximal independent subgraph G_0 of G.
- For any $uv \in E(G) \setminus E(G_0)$ there is a unique stress matrix Ω_{uv} supported on $E(G_0) + uv$ such that $\Omega_{uv} = 1$.
- $\{\Omega_{uv}, uv \in E(G) \setminus E(G_0)\}$ forms a basis of the space of stress matrices of (G, p). I will call these the **fundamental stresses** of (G, p) with respect to G_0.

Back to the theorem – why is it true?

Theorem.

If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

- When $d = 1$, this says that minimally 2-connected graphs have at most $2n-3$ edges. This is a nice exercise. It was (probably) shown by Mader in the early '70s.
- When $d = 2$ this says that minimally globally rigid graphs in the plane have at most $3n-6$ edges. Jordán (2017) showed this using the constructive characterization of globally rigid graphs in $\mathbb{R}^2$.

In both cases the only tight example is K_{d+2}.

The key lemma

Lemma.

Let $A_1 \ldots, A_k \in \mathbb{R}^{n\times n}$ be matrices, $t_i, i \in \{1, \ldots, k\}$ scalars and suppose that $\sum_{i=1}^k t_i A_i$ has rank r. Then there is a subset $I \subseteq \{1, \ldots, k\}$ of size at most r and scalars $t_i', i \in I$ such that $\sum_{i\in I} t_i' A_i$ has rank at least r.

Proving the theorem from the lemma

- Suppose that G is globally rigid with $m > r_d(K_n) + n - d - 1$ edges. Take any minimally rigid spanning subgraph G_0 and a generic realization (G, p). Let $\Omega_1, \dots, \Omega_k$ be the fundamental stresses of (G, p) w.r.t. G_0. Note that $k > n - d - 1$.
- By the stress matrix theorem, there is a linear combination $\sum_{i=1}^{k} t_i \Omega_i$ that has rank $n - d - 1$. Hence by the lemma there is some matrix $\sum_{i \in I} t'_i \Omega_i$ of rank $n - d - 1$ that uses at most $n - d - 1$ of the fundamental stresses.
- This is a stress matrix of a subgraph G' on at most $r_d(K_n) + n - d - 1$ edges. By the stress matrix theorem again, G' is globally rigid. This means that G was not minimally globally rigid.

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Proving the lemma

The main idea:

- (By passing to a submatrix we may assume that $r = n$.)
- Consider the function $f : \mathbb{R}^k \to \mathbb{R}$ defined by

$$(x_1, \ldots, x_k) \mapsto \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

  This is a polynomial of degree at most n. It is not identically zero, since $f(t_1, \ldots, t_k) \neq 0$.
- The lemma follows from looking at f hard enough. (We do not need to look too hard.)

Remark: the discrete version

The "discrete version" of the lemma is also true:

Lemma. (G., 2023+)
Let $A_1 \dots, A_k \in \mathbb{R}^{n\times n}$ be matrices such that $\sum_{i=1}^k A_i$ has rank r. Then there is a subset $I \subseteq \{1, \dots, k\}$ of size at most r such that $\sum_{i\in I} A_i$ has rank at least r.

(Does this have any applications in rigidity theory?)

Tightness and minimum degree

Theorem. (G., Jordán, 2022)
The only minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and with $m = n(d+1) - \binom{d+2}{2}$ edges is K_{d+2}.

Theorem. (G., Jordán, 2022)
If G is a minimally globally rigid graph in $\mathbb{R}^d$, then G has a vertex of degree at most $2d+1$.

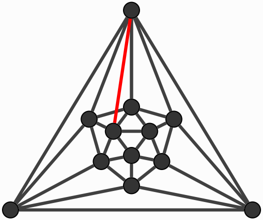

Figure 1: The graph of the icosahedron braced with one edge is minimally globally rigid in $\mathbb{R}^3$ with minimum degree $5 = 2d - 1$.

The “real” explanation?

Let’s return to the interpretation

$$n(d+1) - \binom{d+2}{2} = r_{d+1}(K_n).$$

Is there a connection between global rigidity in $\mathbb{R}^d$ and rigidity in $\mathbb{R}^{d+1}$?

The “real” explanation?

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Is there a connection between global rigidity in $\mathbb{R}^d$ and rigidity in $\mathbb{R}^{d+1}$?

There is!

Theorem. (Jordán, 2017)
If G is rigid in $\mathbb{R}^{d+1}$, then it is globally rigid in $\mathbb{R}^d$.

The globally linked conjecture

Theorem. (Jordán, 2017)
If G is rigid in $\mathbb{R}^{d+1}$, then it is globally rigid in $\mathbb{R}^d$.

Conjecture. (G., Jordán, 2022)
If a pair of vertices $\{u, v\}$ is linked in G in $\mathbb{R}^{d+1}$, then it is globally linked in G in $\mathbb{R}^d$.

The pair $\{u, v\}$ is **linked** (**globally linked**, resp.) in G in $\mathbb{R}^d$ if for every generic realization (G, p) in $\mathbb{R}^d$, the set

$$\{||q(u) - q(v)|| : (G, q) \text{ is equivalent to } (G, p)\}$$

is finite (a singleton, resp.).

A partial result and an interesting consequence

Conjecture. (G., Jordán, 2022)
If a pair of vertices $\{u, v\}$ is linked in G in $\mathbb{R}^{d+1}$, then it is globally linked in G in $\mathbb{R}^d$.

This would imply that minimally globally rigid graphs in $\mathbb{R}^d$ are independent in $\mathbb{R}^{d+1}$. It would follow that such graphs are not only sparse, but everywhere sparse.

Theorem. (G., Jordán, 2022)
The conjecture is true for $d = 1, 2$.

Yet another interpretation of the bound

Recall: the bound in the theorem was

$$n(d+1) - \binom{d+2}{2} = r_d(K_n) + n - d - 1.$$

We interpreted $n-d-1$ as the maximum rank of stress matrices at generic realizations.

The GHT paper also gives a different interpretation in terms of the **shared stress kernel**. Using this notion I can give a different proof of our theorem.

A very recent result

Theorem. (G., 2023+)
If G is minimally globally rigid in $\mathbb{R}^d$, then for each subset $X \subseteq V(G)$ of vertices with $|X| \geq d+1$ we have

$$|E(X)| \leq r_d(G[X]) + |X| - d - 1 \leq (d+1)|X| - \binom{d+2}{2}.$$

Tibor Jordán and Soma Villányi recently also gave a (completely different) proof of this for the subsets X such that $G[X]$ is rigid in $\mathbb{R}^d$.

To recap

Theorem.
If G is a minimally globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices and m edges, then $m \leq n(d+1) - \binom{d+2}{2}$.

- Using the stress matrix theorem + linear algebra we could also characterize tightness, but we could not prove everywhere sparsity.
- There is a conjectured generalization which would also imply everywhere sparsity. It does not seem to help in characterizing tightness.
- Using the notion of a shared stress kernel + algebraic geometry I can prove everywhere sparsity, but cannot characterize tightness.