Algebraic matroids: the combinatorics of (finite) solvability

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Outline

Outline of this talk:

- What is an algebraic matroid?
- Where can we find algebraic matroids “in nature”?
  $\rightarrow$ “Combinatorics of (finite) solvability.”
- Is there a (nice) “combinatorics of unique solvability”?
  $\rightarrow$ Not really.

(conference submission:
“Algebraic realizations of pairs of closure operators”)

I will assume familiarity with the basic notions of matroid theory.

What is an algebraic matroid?

Linear matroid:

representation by vectors in a vector space

+

linear independence.

Algebraic matroid:

representation by elements in a field extension

+

algebraic independence.

For example, take the polynomials $\{x^2, y^2, xy^2, x^2 + y^2\}$ over $\mathbb{Q}$.

What is algebraic (in)dependence?

Let $K \subseteq L$ be fields and $f_1, \ldots, f_m \in L$. We say that $\{f_1, \ldots, f_n\}$ is **algebraically dependent** over K if there is some nonzero polynomial $G \in K[t_1, \ldots, t_m]$ such that $G(f_1, \ldots, f_m) = 0$.

For example:

- $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is algebraically dependent over $\mathbb{Q}$: $\sqrt{2}\sqrt{3} - \sqrt{6} = 0$, so they satisfy $G(t_1, t_2, t_3) = t_1 t_2 - t_3$.
- $\{x^2, y^2, xy^2\}$ is algebraically dependent over $\mathbb{Q}/\mathbb{R}/\mathbb{C}$: $x^2(y^2)^2 - (xy^2)^2 = 0$, so they satisfy $G(t_1, t_2, t_3) = t_1 t_2^2 - t_3^2$.

If $\{f_1, \ldots, f_m\}$ is not algebraically dependent over K, then it is **algebraically independent** over K.

Definitions, examples and facts

The algebraic matroid corresponding to $\{f_1, \ldots, f_m\}$ over K is the matroid on $\{1, \ldots, m\}$ where a subset I is independent if and only if $\{f_i : i \in I\}$ is algebraically independent over K.

A matroid is algebraic over K if it is isomorphic to some matroid of the above form.

- The matroid corresponding to $\{x^2, y^2, xy^2, x^2 + y^2\}$ over $\mathbb{Q}$ is $U_{2,4}$.
- Linear representation $\rightarrow$ algebraic representation by linear forms.
- Over fields of characteristic zero (e.g., $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) every algebraic matroid is also linear.

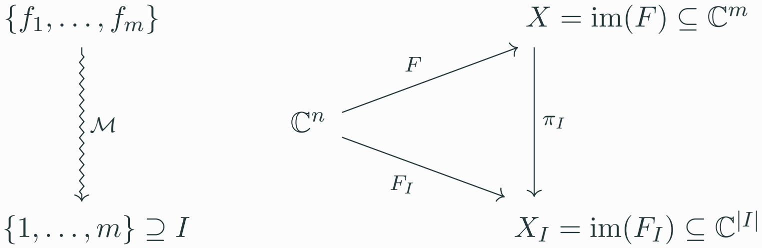
… then what is the point of studying algebraic matroids?

The point

- In some cases the algebraic representation is the “natural” one.
- In some cases the algebraic representation carries interesting additional information.
- (Also, over fields of positive characteristic (e.g., $\mathbb{F}_p$) the situation is much more complicated/interesting.)

The geometric picture – the setup

- Let $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials and let $\mathcal{M}$ be the algebraic matroid on $\{1, \ldots, m\}$ corresponding to $\{f_1, \ldots, f_m\}$ over $\mathbb{C}$.
- Let $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map consisting of these polynomials.
- For $I \subseteq \{1, \ldots, m\}$, let $F_I = (f_i : i \in I) : \mathbb{C}^n \mapsto \mathbb{C}^{|I|}$.



The geometric picture – correspondences

$$\{f_1, \ldots, f_m\} \overset{\mathcal{M}}{\rightsquigarrow} \{1, \ldots, m\} \supseteq I$$

$$\begin{array}{ccc} & & X \subseteq \mathbb{C}^m \\ & \overset{F}{\nearrow} & \downarrow \pi_I \\ \mathbb{C}^n & & \\ & \underset{F_I}{\searrow} & \\ & & X_I \subseteq \mathbb{C}^{|I|} \end{array}$$

I is a spanning set in $\mathcal{M} \iff$ for "almost all" $x \in X_I$, the equation $\pi_I(y) = x$ has finitely many solutions.

I is an independent set in $\mathcal{M} \iff$ for "almost all" $x \in \mathbb{C}^{|I|}$, the equation $\pi_I(y) = x$ has a solution.

Examples

$$
\begin{array}{c}
\{f_1, \ldots, f_m\} \\
\Big\downarrow \rlap{\scriptstyle \mathcal{M}} \\
\{1, \ldots, m\} \supseteq I
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & X \subseteq \mathbb{C}^m \\
 & \nearrow^{F} & \Big\downarrow \rlap{\scriptstyle \pi_I} \\
\mathbb{C}^n & \searrow_{F_I} & \\
 & & X_I \subseteq \mathbb{C}^{|I|}
\end{array}
$$

I will describe two examples of this phenomenon:

- rigidity theory,
- low-rank matrix completion.

Examples – graph rigidity

Let $G = (V, E)$ be a graph, $|V| = n, |E| = m$.

A d-dimensional **framework** is a pair (G, p), where $p : V \to \mathbb{R}^d$ (or equivalently, $p \in \mathbb{R}^{nd}$).

Two frameworks (G, p) and (G, q) are **congruent** if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\|, \quad \forall u, v \in V.$$

A framework (G, p) is **rigid** if there are only finitely many congruence classes of frameworks (G, q) such that the length of each edge of G is the same in (G, p) and (G, q).

Examples – graph rigidity

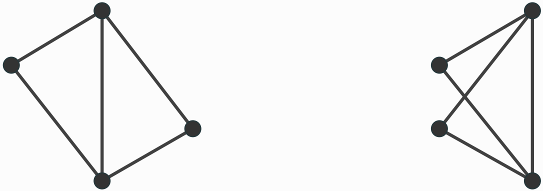


Figure 1: Rigid frameworks for which there are two congruence classes of frameworks with the same vector of edge lengths.

Examples – graph rigidity

For $u, v \in V$, let us define a polynomial function $m_{uv} : \mathbb{R}^{nd} \to \mathbb{R}$ by

$$m_{uv}(p) = \|p(u) - p(v)\|^2 = \sum_{i=1}^{d} (p(u)_i - p(v)_i)^2,$$

and let

$$m_{d,G} : (m_{uv}, uv \in E) : \mathbb{R}^{nd} \to \mathbb{R}^m.$$

- Two frameworks (G, p) and (G, q) are congruent if and only if $m_{d,K_V}(p) = m_{d,K_V}(q)$.
- A framework (G, p) is rigid if and only if

$$\{m_{d,K_V}(q) : q \in \mathbb{R}^{nd}, m_{d,G}(q) = m_{d,G}(p)\}$$

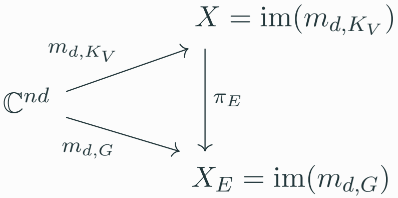
is finite.

Examples – graph rigidity

$$\begin{array}{ccc} & \overset{m_{d,K_V}}{\nearrow} & X = \operatorname{im}(m_{d,K_V}) \\ \mathbb{R}^{nd} & & \downarrow \pi_E \\ & \underset{m_{d,G}}{\searrow} & X_E = \operatorname{im}(m_{d,G}) \end{array}$$

This looks almost like our geometric picture from before.

Examples – graph rigidity



The algebraic matroid corresponding to this picture is the generic d-**dimensional rigidity matroid** $\mathcal{R}_d(K_V)$. Translated back to algebra, this is the algebraic matroid corresponding to the “distance polynomials” $\{m_{uv}, uv \in E(K_V)\}$ over $\mathbb{C}$.

A graph $G = (V, E)$ is **rigid in** $\mathbb{R}^d$ if E is a spanning set in $\mathcal{R}_d(K_V)$. This means that “almost all” d-dimensional frameworks (G, p) are rigid.

Examples – low-rank matrix completion

Let r be an integer and $A \in \mathbb{C}^{n \times n}$ a matrix that is only partially filled. Can the rest of the elements of A be filled in so that the resulting matrix A' has rank at most r?

The entries of an $n \times n$ matrix correspond to the edges of $K_{n,n}$. The set of filled entries in A correspond to some subset $E \subseteq E(K_{n,n})$.

Examples – low-rank matrix completion

Let $X \subseteq \mathbb{C}^{n\times n}$ be the set of matrices of rank at most r, and W_E the set of partially filled matrices that can be completed to a matrix of rank at most r. Then we have the following picture.

$$\begin{array}{c} X \subseteq \mathbb{C}^{n\times n} \\ \Big\downarrow \pi_E \\ X_E \subseteq \mathbb{C}^{|E|} \end{array}$$

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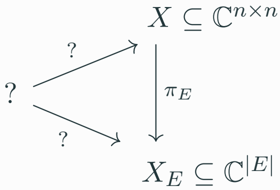
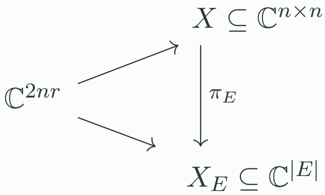


To complete this picture, we can use the fact that a matrix $A \in \mathbb{C}^{n \times n}$ has rank at most r if and only if it can be written as a product $A = BC^T$ with $B, C \in \mathbb{C}^{n \times r}$.

Examples – low-rank matrix completion



This picture encodes an algebraic matroid $\mathcal{C}_r(K_{n,n})$; this is the rank-r matrix completion matroid.

A subset $E \subseteq E(K_{n,n})$ is independent in $\mathcal{C}_r(K_{n,n})$, if "almost all" partially filled matrices where E corresponds to the set of known entries can be completed to a full matrix of rank at most r.

$\mathcal{C}_1(K_{n,n})$ is the graphic matroid of $K_{n,n}$. For $r \geq 2$, no good characterization of $\mathcal{C}_r(K_{n,n})$ is known.

Unique solvability

In both of these examples, there is also a natural “unique solvability” problem.

- A d-dimensional framework (G, p) is globally rigid if the only frameworks (G, q) that have the same edge lengths are the ones congruent to (G, p). A graph $G = (V, E)$ is globally rigid in $\mathbb{R}^d$ if “almost all” d-dimensional frameworks (G, p) are globally rigid.
- A bipartite graph $G = ([n], [n], E)$ is uniquely completable to rank r if “almost all” partially filled matrices where E corresponds to the set of known entries have a unique completion to a matrix of rank at most r.

Combinatorics of unique solvability?

$$I \text{ is spanning in } \mathcal{M} \iff \begin{array}{c}\text{for "almost all" } x \in X_I, \\ \text{the equation } \pi_I(y) = x \\ \text{has finitely many solutions.}\end{array}$$

$$I \text{ is ??? in } \mathcal{M} \iff \begin{array}{c}\text{for "almost all" } x \in X_I, \\ \text{the equation } \pi_I(y) = x \\ \text{has a unique solution.}\end{array}$$

Combinatorics of unique solvability?

I is spanning in $\mathcal{M}$ $\iff$ for "almost all" $x \in X_I$, the equation $\pi_I(y) = x$ has finitely many solutions.

I is strongly spanning "in $\mathcal{M}$" $\iff$ for "almost all" $x \in X_I$, the equation $\pi_I(y) = x$ has a unique solution.

But strongly spanning sets are determined by the representation, not by the matroid!

Closure and strong closure

Strongly spanning sets can be defined in any algebraic representation M (over any field K).

We can go one step further and define a closure operator corresponding to strongly spanning sets: the **"strong closure"** $\mathrm{scl}_K^M(I)$ of a subset I is the largest subset of the ground set in which I is strongly spanning.

We also have the usual closure operator cl_K^M of the matroid corresponding to the representation.

Question: given cl_K^M (i.e., the matroid structure) what can we say about scl_K^M (i.e., the combinatorical structure of strongly spanning sets)?

The mixed exchange property

Let $\mathrm{cl} = \mathrm{cl}_K^M$ and $\mathrm{scl} = \mathrm{scl}_K^M$.

Since cl is the closure operator of a matroid, it satisfies the *Mac Lane-Steinitz exchange property*:

If $x, y \notin \mathrm{cl}(I)$ and $x \in \mathrm{cl}(I + y)$, then $y \in \mathrm{cl}(I + x)$,

for any subset I of the ground set.

In general, scl does not have this property!

But cl and scl also satisfy the following "mixed exchange property":

If $x \notin \mathrm{scl}(I), y \notin \mathrm{cl}(I)$ and $x \in \mathrm{scl}(I + y)$, then $y \in \mathrm{cl}(I + x)$.

The edge cases

Are there any other combinatorial conditions that scl must satisfy? We may look at the edge cases: algebraic representations where scl is the “smallest” and “largest” possible.

Theorem. Any matroid that is algebraic over a field K has an algebraic representation M for which scl_K^M is the identity map on M.

Theorem. If a matroid is linear over a field K, then it has an algebraic representation M for which $\mathrm{scl}_K^M = \mathrm{cl}_K^M$. In particular, if K has characteristic zero, then every algebraic matroid over K has such a representation.

It is unclear whether the same is true for algebraic matroids that are not linear (over a field of positive characteristic).

Main takeaways

- Algebraic matroids (over algebraically closed fields) have a nice geometric interpretation.
- Whenever we see a particular kind of "geometric picture", we can suspect that there is an algebraic matroid in the background.
- Algebraic representations also carry the additional structure of strongly spanning sets and the strong closure, but in full generality these do not seem to lead to an interesting combinatorial theory.