New necessary conditions for global rigidity

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Two new results

If G is globally rigid in $\mathbb{R}^d$ on at least $d+2$ vertices, then

- it is (rigid and) M-connected in $\mathbb{R}^d$;
  ($\Longrightarrow$ it is redundantly rigid in $\mathbb{R}^d$ = Hendrickson's theorem)
- (if $d \geq 2$) it is fully reconstructible in $\mathbb{C}^d$.
  ($\Longrightarrow$ it is strongly reconstructible in $\mathbb{R}^d$ = a result of Gortler, Theran and Thurston)

Preprint:

G., Steven J. Gortler, Tibor Jordán,
Globally rigid graphs are fully reconstructible,
arXiv:2105.04363

Outline

Main theme: three “languages” of generic (global) rigidity:

- combinatorics – the rigidity matroid, globally rigid graphs;
- linear algebra – the rigidity matrix, stresses and infinitesimal motions;
- geometry – the measurement variety.

Outline of this talk:

- a long setup;
- M-connected and M-separable graphs;
- unlabeled reconstructibility.

Setting up

Setting up – Frameworks

Let $G = (V, E)$ be a graph on n vertices. A **framework** in $\mathbb{R}^d$ is a pair (G, p) where $p : V \to \mathbb{R}^d$, or equivalently $p \in \mathbb{R}^{nd}$. We also say that (G, p) is a **realization** of G in $\mathbb{R}^d$. It is **generic** if the coordinates of $p \in \mathbb{R}^{nd}$ are algebraically independent over $\mathbb{Q}$.

For $u, v \in V$, let

$$m_{uv}(p) = \sum_{i=1}^{d} (p(u)_i - p(v)_i)^2,$$

and define $m_{d,G} : \mathbb{R}^{nd} \to \mathbb{R}^E$ by

$$m_{d,G}(p) = (m_{uv}(p))_{uv \in E}\,.$$

Setting up – Equivalence and congruence

We say that two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are **equivalent** if

$$m_{d,G}(p) = m_{d,G}(q).$$

They are **congruent** if

$$m_{d,K_V}(p) = m_{d,K_V}(q),$$

where K_V denotes the complete graph on vertex set V.

Setting up – Rigidity

A framework (G, p) in $\mathbb{R}^d$ is rigid if every equivalent framework (G, q) in $\mathbb{R}^d$ that is sufficiently close to (G, p) is congruent to it. Equivalently, (G, p) is rigid if every continuous motion of the vertices that preserves the edge lengths consists of congruent frameworks.

Theorem. (Asimow and Roth 1978)
If some generic realization (G, p) in $\mathbb{R}^d$ is rigid, then every generic realization of G in $\mathbb{R}^d$ is rigid.

The graph G is rigid in $\mathbb{R}^d$ if some (equivalently, if every) generic framework (G, p) in $\mathbb{R}^d$ is rigid.

Setting up – Global rigidity

A framework (G, p) in $\mathbb{R}^d$ is globally rigid if every equivalent framework (G, q) in $\mathbb{R}^d$ is congruent to (G, p).

Theorem. (Connelly 2005, Gortler, Healy and Thurston 2010) If some generic realization (G, p) in $\mathbb{R}^d$ is globally rigid, then every generic realization of G in $\mathbb{R}^d$ is globally rigid.

The graph G is globally rigid in $\mathbb{R}^d$ if some (equivalently, if every) generic framework (G, p) in $\mathbb{R}^d$ is globally rigid.

The rigidity matrix $R(G,p)$ of a framework (G,p) is $(1/2$ times$)$ the Jacobian of $m_{d,G}$ at p. In particular, it is a matrix with rows indexed by the edges of G and columns indexed by the nd coordinates.

Theorem. (Asimow and Roth 1978)
A generic framework (G,p) in $\mathbb{R}^d$ is rigid if and only if

$$\mathrm{rk}(R(G,p)) = nd - \binom{d+1}{2},$$

where n denotes the number of vertices of G.

The rank $r_d(G)$ of G is $\mathrm{rk}(R(G,p))$ for any generic realization (G,p) in $\mathbb{R}^d$.

The infinitesimal motions of (G,p) are the members of $\ker(R(G,p))$, i.e. vectors $q \in \mathbb{R}^{nd}$ such that $R(G,p)q = 0$.

The stresses of (G,p) are the members of $\mathrm{coker}(R(G,p))$, i.e. vectors $\omega \in \mathbb{R}^E$ such that $\omega R(G,p) = 0$.

Setting up – The rigidity matroid

The rows of $R(G, p)$ define a matroid on the edge set of G. We get the same matroid from every generic realization of G in $\mathbb{R}^d$, and this is what we call the **d-dimensional (generic) rigidity matroid** of G. We denote it by $\mathcal{R}_d(G)$.

We say that $G = (V, E)$ is **M-independent** if E is independent in $\mathcal{R}_d(G)$, that is, if the rows of $R(G, p)$ are linearly independent. It is an **M-circuit** if it is not M-independent but every proper subgraph is.

Setting up – Linear representation

Combinatorial	Linear algebra	Geometry
Rigidity matroid	linear: $R(G,p)$ for generic (G,p)	
Rank function	$\mathrm{rk}(R(G,p))$	
M-independent	stress-free	
M-circuit	unique stress ω with $\omega_e \neq 0 \quad \forall e \in E$	
Globally rigid	*	

Setting up – Real vs. complex rigidity

All of the above generalizes to complex frameworks.

Let $G = (V, E)$ be a graph, and $u, v \in V$ a pair of vertices. We extend $m_{d,G}$ to a $\mathbb{C}^{nd} \to \mathbb{C}^E$ function by

$$m_{d,G}(p) = \left(\sum_{i=1}^{d} (p(u)_i - p(v)_i)^2 \right)_{uv \in E}.$$

Equivalence, congruence, rigidity and global rigidity can be defined analogously to the real case using $m_{d,G}$.

Theorem. (Gortler, Theran and Thurston 2019, Gortler and Thurston 2014)
A graph is rigid (globally rigid) in $\mathbb{R}^d$ if and only if it is rigid (globally rigid) in $\mathbb{C}^d$.

Setting up – The measurement variety

A subset $X \subseteq \mathbb{C}^E$ is an **(affine) variety** if there are polynomials $f_1, \ldots, f_k \in \mathbb{C}[x_e : e \in E]$ such that

$$X = V(f_1, \ldots, f_k) = \{x \in \mathbb{C}^E : f_1(x) = \ldots = f_k(x) = 0\}.$$

Let G be a graph and $d \geq 1$. The (d-dimensional) **measurement variety** of G is the smallest variety in $\mathbb{C}^E$ that contains $m_{d,G}(\mathbb{C}^{nd})$. It is denoted by $M_{d,G}$.

(In other words, $M_{d,G}$ is the Zariski-closure of $m_{d,G}(\mathbb{C}^{nd})$.)

Combinatorial	Linear algebra	Geometry
Rigidity matroid	linear: $R(G,p)$ for generic (G,p)	algebraic: $M_{d,G}$
Rank function	$\mathrm{rk}(R(G,p))$	$\dim(M_{d,G})$
M-independent	stress-free	$M_{d,G} = \mathbb{C}^E$
M-circuit	unique stress ω with $\omega_e \neq 0 \quad \forall e \in E$	$M_{d,G} = V(f)$
Globally rigid	*	?

M-connected and M-separable graphs

M-connectivity – Connected and separable matroids

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. A separation of $\mathcal{M}$ is a non-trivial partition (E_1, E_2) of E such that for every $I \subseteq E$,

$$I \text{ is independent} \iff I \cap E_1 \text{ and } I \cap E_2 \text{ are independent.}$$

This is equivalent to

$$r_{\mathcal{M}}(E) = r_{\mathcal{M}}(E_1) + r_{\mathcal{M}}(E_2).$$

We say that $\mathcal{M}$ is separable if it has a separation and connected otherwise.

M-connectivity – M-connected graphs

A graph G is **M-connected in $\mathbb{R}^d$** if $\mathcal{R}_d(G)$ is connected.
Otherwise G is **M-separable in $\mathbb{R}^d$**.

- In $\mathbb{R}^1$, M-connected $\Leftrightarrow$ globally rigid $\Leftrightarrow$ 2-connected.
- In $\mathbb{R}^2$, globally rigid $\Rightarrow$ M-connected $\Rightarrow$ redundantly rigid.
- In $\mathbb{R}^d, d \geq 3$, globally rigid $\overset{?}{\Rightarrow}$ M-connected $\not\Rightarrow$ rigid.

M-connectivity – Hendrickson's theorem

Theorem. (Hendrickson 1992)
If G has at least $d+2$ vertices and is globally rigid in $\mathbb{R}^d$, then it is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.

Redundant rigidity $\Leftrightarrow$ $G-e$ is rigid for every edge e
$\Leftrightarrow$ G is rigid and $\mathcal{R}_d(G)$ has no separations of the form $(\{e\}, E-e)$.

So it is natural to ask whether globally rigid graphs (on at least $d+2$ vertices) are M-connected in $\mathbb{R}^d$.

M-connectivity – Translating separability

Let $G = (V, E)$ be a graph, (E_1, E_2) a non-trivial partition of E and let $G_i = (V, E_i), i = 1, 2.$ Then the following are equivalent.

- (E_1, E_2) is a separation of $\mathcal{R}_d(G)$.
- For any generic framework (G, p) in $\mathbb{C}^d$ we have

$$\operatorname{coker}(R(G,p)) = \operatorname{coker}(R(G_1,p)) \oplus \operatorname{coker}(R(G_2,p)).$$

- (GGJ, Lemma 3.1) We have $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$ (under the identification $\mathbb{C}^E = \mathbb{C}^{E_1} \times \mathbb{C}^{E_2}$).

Theorem. (Connelly 2005; Gortler, Healy and Thurston 2010)
Let G be globally rigid in $\mathbb{R}^d$. Then every generic framework (G, p) has a stress ω such that for any realization (G, q), if ω is also a stress of (G, q), then (G, q) is an affine image of (G, p).

(That is, there is a matrix $A \in \mathbb{R}^{d \times d}$ and a vector $b \in \mathbb{R}^d$ such that $q(v) = Ap(v) + b$ for every vertex $v \in V$.)

M-connectivity – Globally rigid graphs are M-connected

Theorem. (GGJ, Theorem 3.5)
Let $G = (V, E)$ be a graph on at least $d + 2$ vertices. If G is globally rigid in $\mathbb{R}^d$, then it is M-connected in $\mathbb{R}^d$.

Proof sketch:

- Suppose that G has a separation (E_1, E_2) and let $G_i = (V, E_i)$. Let (G, p) be a generic realization in $\mathbb{C}^d$.
- Now $m_{d,G}(p) = (x_1, x_2) \in M_{d,G_1} \times M_{d,G_2} = M_{d,G}$.
- Since $m_{d,G_2}(2p) = 4x_2$, we also have that $(x_1, 4x_2) \in M_{d,G}$. It turns out that there is a generic realization (G, q) in $\mathbb{C}^d$ with $m_{d,G}(q) = (x_1, 4x_2)$.
- We can show that (G, q) is not an affine image of (G, p) but it satisfies the same stresses, contradicting global rigidity.

Unlabeled reconstruction

Unlabeled reconstruction – Comparing graphs

We want to examine the situation when $m_{d,G}(p) = m_{d,H}(q)$ for a pair of frameworks (G, p) and (H, q), where $G = (V, E)$ and $H = (V', E')$.

What does this mean?

It means that there is a bijection $\psi : E \to E'$, inducing a bijection $\Psi : \mathbb{C}^E \to \mathbb{C}^{E'}$ such that $m_{d,H}(q) = \Psi(m_{d,G}(p))$.

We shall say that $m_{d,G}(p) = m_{d,G}(q)$ **under the edge bijection ψ**. Similarly, we can have $M_{d,G} = M_{d,H}$ under the edge bijection ψ.

Unlabeled reconstruction – The big result

Theorem. (Gortler, Theran and Thurston, 2019)
Fix $d \geq 2$ and let (G, p) be a generic framework in $\mathbb{R}^d$, where G is globally rigid in $\mathbb{R}^d$ on $n \geq d + 2$ vertices. Suppose that there is another framework (H, q), with H also on n vertices and such that $m_{d,G}(p) = m_{d,H}(q)$ under some edge bijection $\psi : E(G) \to E(H)$.

Then there is a graph isomorphism $\varphi : V(G) \to V(H)$ which induces ψ, that is, for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$. In particular, G and H are isomorphic and the frameworks (G, p) and (H, q) are congruent after relabeling, i.e. (G, p) is congruent to $(G, q \circ \varphi)$.

Unlabeled reconstruction – What can go wrong?

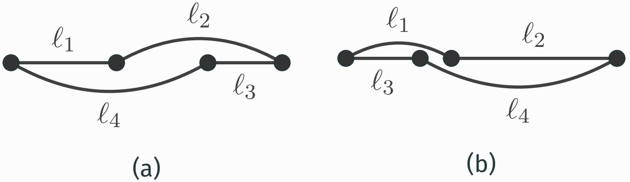

Figure 1: Two generic realizations of C_4 in $\mathbb{R}^1$ with the same edge lengths, where the mapping between the corresponding edges does not arise from a graph isomorphism.

Unlabeled reconstruction – What can go wrong?

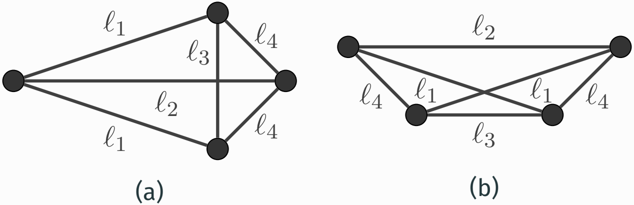

Figure 2: Two non-generic, non-congruent realizations of K_4 with the same edge lengths. We can obtain these realizations in $\mathbb{R}^2$ by putting $\ell_1 = \sqrt{10}, \ell_2 = 4, \ell_3 = 2$.

This example is from (Boutin and Kemper 2004), who proved the above unlabeled reconstruction theorem in the case of complete graphs.

Unlabeled reconstruction – Strong reconstructibility

A graph G is **strongly reconstructible** in $\mathbb{C}^d$ if whenever $M_{d,G} = M_{d,H}$ under an edge bijection $\psi : E(G) \to E(H)$ for some graph H on the same number of vertices as G, there is a graph isomorphism $\varphi : V(G) \to V(H)$ that induces ψ.

(That is, for any edge $uv \in E(G), \psi(uv) = \varphi(u)\varphi(v)$.)

Theorem. (Gortler, Theran and Thurston 2019)
Fix $d \geq 2$ and let G be a graph on $n \geq d + 2$ vertices. If G is globally rigid in $\mathbb{R}^d$, then it is strongly reconstructible in $\mathbb{C}^d$.

Unlabeled reconstruction – Strong reconstructibility

Theorem. (Gortler, Theran and Thurston 2019)
Fix $d \geq 2$ and let G be a graph on $n \geq d+2$ vertices. If G is globally rigid in $\mathbb{R}^d$, then it is strongly reconstructible in $\mathbb{C}^d$.

Proof idea:

- If $M_{d,G} = M_{d,H}$ and H also has n vertices, then H is globally rigid in $\mathbb{R}^d$ as well.
- Using global rigidity, we can prove $M_{d-1,G} = M_{d-1,H}$; iterating, we get $M_{1,G} = M_{1,H}$.
- $M_{1,G}$ "encodes" $\mathcal{R}_1(G)$, which is just the graphic matroid. But G is 3-connected (from global rigidity), and it is known that 3-connected graphs without isolated vertices are uniquely determined by their graphic matroid.

Unlabeled reconstruction – Full reconstructibility

A graph G without isolated vertices is fully reconstructible in $\mathbb{C}^d$ if whenever $M_{d,G} = M_{d,H}$ under an edge bijection $\psi : E(H) \to E(G)$ for some graph H without isolated vertices, there is a graph isomorphism $\varphi : V(H) \to V(G)$ that induces ψ.

Theorem. (GGJ, Theorem 3.6)
Fix $d \geq 2$ and let G be a graph on $n \geq d + 2$ vertices. If G is globally rigid in $\mathbb{R}^d$, then it is fully reconstructible in $\mathbb{C}^d$.

(The $d = 2$ case was already proved in (G. and Jordán 2021))

Unlabeled reconstruction – Full reconstructibility

Proof idea:

- Suppose that $M_{d,G} = M_{d,H}$. Let n' denote the number of vertices of H. If $n = n'$, then we are done by the previous theorem.
- Otherwise the equality of dimensions implies that $n' > n$ and H is a flexible graph.
- Using the global rigidity of G we can prove $M_{d-1,H} \subseteq M_{d-1,G}$.
- From this, we get bounds on the generic dimension of infinitesimal motions of H in d and $d-1$ motions. We show that as we decrease d, this generic dimension cannot decrease "too much". But in one dimension it must be 1, a contradiction.

Unlabeled reconstruction – M-connectivity, again

Theorem. (GGJ, Theorem 3.7)
Let G be a graph without isolated vertices and suppose that G is fully reconstructible in $\mathbb{C}^d$. Then it is M-connected in $\mathbb{R}^d$.

Proof idea: If G is M-separable, then $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$ for some subgraphs G_1 and G_2. But the disjoint union of G_1 and G_2 also has the same measurement variety.

Other results

Also in the preprint:

- new examples of H-graphs,
- new (counter-)examples of strongly and fully reconstructible graphs,
- some results on M-connected and M-separable graphs in $\mathbb{R}^d$.

An M-separable graph in $\mathbb{R}^3$